Exercise 2

In Exercises 1 through 4, take the indented contour in Fig. 101 (Sec. 82).

Evaluate the improper integral

\[ \int_{0}^{\infty} \frac{x^a}{(x^2 + 1)^2} \, dx, \quad \text{where} \quad -1 < a < 3 \text{ and } x^a = \exp(a \ln x). \]

**Ans.** \( \frac{(1 - a)\pi}{4 \cos(a\pi/2)} \)

Solution

In order to evaluate the integral, consider the corresponding function in the complex plane, \( \frac{z^a}{(z^2 + 1)^2} \), and the contour in Fig. 101. Singularities occur where the denominator is equal to zero.

\[
(z^2 + 1)^2 = 0 \\
z^2 + 1 = 0 \\
z = \pm i
\]

The singular point of interest to us is the one that lies within the closed contour, \( z = i \). \( z^a \) can be written in terms of the logarithm function as

\[ z^a = \exp(a \log z), \]

so a branch cut for the function has to be chosen. For convenience, we choose it to be the axis of negative imaginary numbers.

\[
z^a = \exp[a(\ln r + i\theta)], \quad \left( |z| > 0, \quad -\frac{\pi}{2} < \theta < \frac{3\pi}{2} \right)
\]

\[ = r^a e^{i\theta}, \]

where \( r = |z| \) is the magnitude of \( z \). According to Cauchy’s residue theorem, the integral of \( \frac{z^a}{(z^2 + 1)^2} \) around the closed contour is equal to \( 2\pi i \) times the sum of the residues at the enclosed singularities.

\[
\oint_{C} \frac{z^a}{(z^2 + 1)^2} \, dz = 2\pi i \operatorname{Res}_{z=i} \frac{z^a}{(z^2 + 1)^2}
\]

This closed loop integral is the sum of four integrals, one over each arc in the loop.

\[
\int_{L_1} \frac{z^a}{(z^2 + 1)^2} \, dz + \int_{C_\rho} \frac{z^a}{(z^2 + 1)^2} \, dz + \int_{L_2} \frac{z^a}{(z^2 + 1)^2} \, dz + \int_{C_R} \frac{z^a}{(z^2 + 1)^2} \, dz = 2\pi i \operatorname{Res}_{z=i} \frac{z^a}{(z^2 + 1)^2}
\]

The parameterizations for the arcs are as follows.

\[
L_1 : \quad z = re^{i\theta}, \quad r = \rho \rightarrow r = R \\
L_2 : \quad z = re^{i\pi}, \quad r = R \rightarrow r = \rho \\
C_\rho : \quad z = \rho e^{i\theta}, \quad \theta = \pi \rightarrow \theta = 0 \\
C_R : \quad z = Re^{i\theta}, \quad \theta = 0 \rightarrow \theta = \pi
\]

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Figure 1: This is Fig. 101 with the singularity at \( z = i \) marked. The squiggly line represents the branch cut \((|z| > 0, \ -\pi/2 < \theta < 3\pi/2)\).

As a result,

\[
2\pi i \operatorname{Res}_{z=i} \frac{z^a}{(z^2 + 1)^2} = \int_{\rho}^{R} \frac{(re^{i0})^a}{[(re^{i0})^2 + 1]^2} (dr \ e^{i0}) + \int_{C_{\rho}} \frac{z^a}{(z^2 + 1)^2} dz + \int_{R}^{\rho} \frac{(re^{i\pi})^a}{[(re^{i\pi})^2 + 1]^2} (dr \ e^{i\pi}) + \int_{C_{R}} \frac{z^a}{(z^2 + 1)^2} dz
\]

\[
= \int_{\rho}^{R} \frac{r^a}{(r^2 + 1)^2} dr + \int_{C_{\rho}} \frac{z^a}{(z^2 + 1)^2} dz + \int_{R}^{\rho} \frac{r^a e^{i\alpha}}{(r^2 + 1)^2} (-dr) + \int_{C_{R}} \frac{z^a}{(z^2 + 1)^2} dz
\]

\[
= \int_{\rho}^{R} \frac{r^a}{(r^2 + 1)^2} dr + \int_{C_{\rho}} \frac{z^a}{(z^2 + 1)^2} dz + \int_{R}^{\rho} \frac{r^a e^{i\alpha}}{(r^2 + 1)^2} dr + \int_{C_{R}} \frac{z^a}{(z^2 + 1)^2} dz
\]

\[
= \int_{\rho}^{R} \frac{r^a + r^a e^{i\alpha}}{(r^2 + 1)^2} dr + \int_{C_{\rho}} \frac{z^a}{(z^2 + 1)^2} dz + \int_{C_{R}} \frac{z^a}{(z^2 + 1)^2} dz
\]

\[
= (1 + e^{i\alpha}) \int_{\rho}^{R} \frac{r^a}{(r^2 + 1)^2} dr + \int_{C_{\rho}} \frac{z^a}{(z^2 + 1)^2} dz + \int_{C_{R}} \frac{z^a}{(z^2 + 1)^2} dz.
\]

Take the limit now as \( \rho \to 0 \) and \( R \to \infty \). The integral over \( C_{\rho} \) tends to zero, and the integral over \( C_{R} \) tends to zero. Proof for these statements will be given at the end.

\[
(1 + e^{i\alpha}) \int_{0}^{\infty} \frac{r^a}{(r^2 + 1)^2} dr = 2\pi i \operatorname{Res}_{z=i} \frac{z^a}{(z^2 + 1)^2}
\]

The denominator can be written as \((z^2 + 1)^2 = (z + i)^2 (z - i)^2\). From this we see that the multiplicity of the factor \( z - i \) is 2. The residue at \( z = i \) can then be calculated by

\[
\operatorname{Res}_{z=i} \frac{z^a}{(z^2 + 1)^2} = \frac{\phi'(i)}{(2 - 1)!} = \phi'(i),
\]

where \( \phi(z) \) is the same function as the integrand without \((z - i)^2\).

\[
\phi(z) = \frac{z^a}{(z + i)^2}
\]

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Calculate its derivative using the quotient rule.

\[
\phi'(z) = \frac{az^{a-1}(z + i)^2 - 2(z + i)z^a}{(z + i)^4} = \frac{az^{a-1}(z + i) - 2z^a}{(z + i)^3}
\]

So then

\[
\phi'(i) = \frac{ai^{a-1}(2i) - 2i^a}{(2i)^3} = \frac{2ai^a - 2i^a}{8i^3} = \frac{2i^a(a - 1)}{-8i} = \frac{i^a(1 - a)}{4i}
\]

and

\[
\text{Res}_{z=i} \frac{z^a}{(z^2 + 1)^2} = \frac{i^a(1 - a)}{4i}.
\]

Consequently,

\[
(1 + e^{ia\pi}) \int_0^\infty \frac{r^a}{(r^2 + 1)^2} dr = 2\pi i \left[ \frac{i^a(1 - a)}{4i} \right] = \frac{i^a(1 - a)\pi}{2}.
\]

Use the fact that \(i = e^{i\pi/2}\) to write \(i^a = e^{ia\pi/2}\).

\[
(1 + e^{ia\pi}) \int_0^\infty \frac{r^a}{(r^2 + 1)^2} dr = \frac{e^{ia\pi/2}(1 - a)\pi}{2}
\]

Divide both sides by \(1 + e^{ia\pi}\).

\[
\int_0^\infty \frac{r^a}{(r^2 + 1)^2} dr = \frac{(1 - a)\pi}{2} \frac{e^{ia\pi/2}}{1 + e^{ia\pi}} = \frac{(1 - a)\pi}{2} \frac{1}{e^{-ia\pi/2} + e^{ia\pi/2}} = \frac{(1 - a)\pi}{2} \frac{1}{2 \cos(a\pi/2)}
\]

Change the dummy variable of integration to \(x\). Therefore,

\[
\int_0^\infty \frac{x^a}{(x^2 + 1)^2} dx = \frac{(1 - a)\pi}{4 \cos(a\pi/2)}
\]
The Integral Over \( C_\rho \)

Our aim here is to show that the integral over \( C_\rho \) tends to zero in the limit as \( \rho \to 0 \). The parameterization of the small semicircular arc in Fig. 101 is \( z = \rho e^{i\theta} \), where \( \theta \) goes from \( \pi \) to 0.

\[
\int_{C_\rho} \frac{z^a}{(z^2 + 1)^2} \, dz = \int_0^\pi \frac{(\rho e^{i\theta})^a}{(\rho^2 e^{2i\theta} + 1)^2} (\rho^2 e^{2i\theta} \, d\theta)
\]

\[
= \int_0^\pi \rho^{1+a} e^{i\theta(1+a)} (i \, d\theta)
\]

In the limit as \( \rho \to 0 \), we have

\[
\lim_{\rho \to 0} \int_{C_\rho} \frac{z^a}{(z^2 + 1)^2} \, dz = \lim_{\rho \to 0} \int_0^\pi \rho^{1+a} e^{i\theta(1+a)} (i \, d\theta).
\]

Because the limits of integration are constant, the limit may be brought inside the integral.

\[
\lim_{\rho \to 0} \int_{C_\rho} \frac{z^a}{(z^2 + 1)^2} \, dz = \int_0^\pi \lim_{\rho \to 0} \rho^{1+a} e^{i\theta(1+a)} (i \, d\theta)
\]

Since \( -1 < a < 3 \), \( 1 + a \) is positive, and \( \rho^{1+a} \) tends to zero as a result. Therefore,

\[
\lim_{\rho \to 0} \int_{C_\rho} \frac{z^a}{(z^2 + 1)^2} \, dz = 0.
\]

The Integral Over \( C_R \)

Our aim here is to show that the integral over \( C_R \) tends to zero in the limit as \( R \to \infty \). The parameterization of the large semicircular arc in Fig. 101 is \( z = Re^{i\theta} \), where \( \theta \) goes from 0 to \( \pi \).

\[
\int_{C_R} \frac{z^a}{(z^2 + 1)^2} \, dz = \int_0^\pi \frac{(Re^{i\theta})^a}{(Re^{2i\theta} + 1)^2} (Re^{i\theta} \, d\theta)
\]

\[
= \int_0^\pi R^{1+a} e^{i\theta(1+a)} (R^2 e^{2i\theta} + 1)^2 (i \, d\theta)
\]

Now consider the integral’s magnitude.

\[
\left| \int_{C_R} \frac{z^a}{(z^2 + 1)^2} \, dz \right| = \left| \int_0^\pi \frac{R^{1+a} e^{i\theta(1+a)}}{(R^2 e^{2i\theta} + 1)^2} (i \, d\theta) \right|
\]

\[
\leq \int_0^\pi \left| \frac{R^{1+a} e^{i\theta(1+a)}}{(R^2 e^{2i\theta} + 1)^2} \right| (i \, d\theta) = \int_0^\pi \frac{R^{1+a} e^{i\theta(1+a)}}{|R^2 e^{2i\theta} + 1|^2} (i \, d\theta)
\]

\[
= \int_0^\pi \frac{R^{1+a}}{|R^2 e^{2i\theta} + 1|^2} (1+i) \, d\theta
\]

\[
= \int_0^\pi \frac{R^{1+a}}{|R^2 e^{2i\theta} + 1|^2} \, d\theta
\]

\[
\leq \int_0^\pi \frac{R^{1+a}}{|R^2 e^{2i\theta} - 1|^2} \, d\theta
\]

\[
= \int_0^\pi \frac{R^{1+a}}{(R^2 - 1)^2} \, d\theta
\]
So we have

\[
\left| \int_{C_R} \frac{z^a}{(z^2 + 1)^2} \, dz \right| \leq \frac{R^{1+a}}{(R^2 - 1)^2} \pi \\
= \frac{R^{1+a}}{R^4 (1 - \frac{1}{R^2})^2} \pi \\
= \frac{R^{3-a}}{R^3 (1 - \frac{1}{R^2})^2}.
\]

Take the limit of both sides as \( R \to \infty \).

\[
\lim_{R \to \infty} \left| \int_{C_R} \frac{z^a}{(z^2 + 1)^2} \, dz \right| \leq \lim_{R \to \infty} \frac{\pi}{R^{3-a} (1 - \frac{1}{R^2})^2}
\]

Since \(-1 < a < 3\), \(3 - a\) is positive, and the denominator tends to infinity as a result.

\[
\lim_{R \to \infty} \left| \int_{C_R} \frac{z^a}{(z^2 + 1)^2} \, dz \right| \leq 0
\]

The magnitude of a number cannot be negative.

\[
\lim_{R \to \infty} \left| \int_{C_R} \frac{z^a}{(z^2 + 1)^2} \, dz \right| = 0
\]

The only number that has a magnitude of zero is zero. Therefore,

\[
\lim_{R \to \infty} \int_{C_R} \frac{z^a}{(z^2 + 1)^2} \, dz = 0.
\]