Exercise 3

In Exercises 1 through 4, take the indented contour in Fig. 101 (Sec. 82).

Use the function

$$f(z) = \frac{z^{1/3}\log z}{z^2+1} = \frac{e^{(1/3)\log z}\log z}{z^2+1} \qquad \left(|z|>0,\ -\frac{\pi}{2}<\arg z<\frac{3\pi}{2}\right)$$

to derive this pair of integration formulas:

$$\int_0^\infty \frac{\sqrt[3]{x}\ln x}{x^2+1}\,dx = \frac{\pi^2}{6}, \qquad \int_0^\infty \frac{\sqrt[3]{x}}{x^2+1}\,dx = \frac{\pi}{\sqrt{3}}.$$

Solution

In order to evaluate these integrals, consider the given function in the complex plane and the contour in Fig. 101. Singularities occur where the denominator is equal to zero.

$$z^2+1=0$$
$$z=\pm i$$

The singular point of interest to us is the one that lies within the closed contour, $z=i$. $z^{1/3}$ can be written in terms of the logarithm function as

$$z^{1/3} = \exp\left(\frac{1}{3}\log z\right),$$

so a branch cut for the function has to be chosen. For convenience, we choose it to be the axis of negative imaginary numbers.

$$\begin{aligned} z^{1/3} &= \exp\left[\frac{1}{3}(\ln r + i\theta)\right], \quad \left(|z|>0,\ -\frac{\pi}{2}<\theta<\frac{3\pi}{2}\right) \\ &= r^{1/3}e^{i\theta/3}, \end{aligned}$$

where $r=|z|$ is the magnitude of z and $\theta=\arg z$ is the argument of z. According to Cauchy's residue theorem, the integral of $z^{1/3}\log z/(z^2+1)$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{z^{1/3}\log z}{z^2+1}\,dz = 2\pi i \operatorname*{Res}_{z=i} \frac{z^{1/3}\log z}{z^2+1}$$

This closed loop integral is the sum of four integrals, one over each arc in the loop.

$$\int_{L_1} \frac{z^{1/3}\log z}{z^2+1}\,dz + \int_{L_2} \frac{z^{1/3}\log z}{z^2+1}\,dz + \int_{C_\rho} \frac{z^{1/3}\log z}{z^2+1}\,dz + \int_{C_R} \frac{z^{1/3}\log z}{z^2+1}\,dz = 2\pi i \operatorname*{Res}_{z=i} \frac{z^{1/3}\log z}{z^2+1}$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L_1: &\quad z = re^{i0}, &\quad r=\rho &\quad \to \quad r=R \\ L_2: &\quad z = re^{i\pi}, &\quad r=R &\quad \to \quad r=\rho \\ C_\rho: &\quad z = \rho e^{i\theta}, &\quad \theta=\pi &\quad \to \quad \theta=0 \\ C_R: &\quad z = Re^{i\theta}, &\quad \theta=0 &\quad \to \quad \theta=\pi \end{aligned}$$

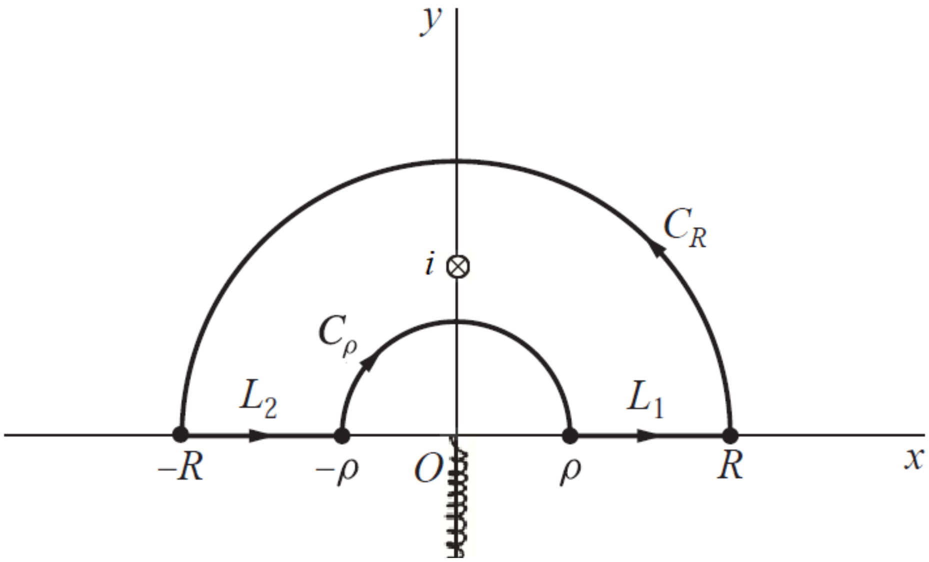

Figure 1: This is Fig. 101 with the singularity at $z = i$ marked. The squiggly line represents the branch cut ($|z| > 0,\ -\pi/2 < \theta < 3\pi/2$).

As a result,

$$\begin{aligned}
2\pi i \operatorname*{Res}_{z=i} \frac{z^{1/3}\log z}{z^2+1} &= \int_\rho^R \frac{(re^{i0})^{1/3}\log(re^{i0})}{(re^{i0})^2+1}(dr\, e^{i0}) + \int_R^\rho \frac{(re^{i\pi})^{1/3}\log(re^{i\pi})}{(re^{i\pi})^2+1}(dr\, e^{i\pi}) + \int_{C_\rho} \frac{z^{1/3}\log z}{z^2+1}\,dz \\
&\quad + \int_{C_R} \frac{z^{1/3}\log z}{z^2+1}\,dz \\
&= \int_\rho^R \frac{r^{1/3}(\ln r + i0)}{r^2+1}\,dr + \int_R^\rho \frac{r^{1/3}e^{i\pi/3}(\ln r + i\pi)}{(-r)^2+1}(-dr) + \int_{C_\rho} \frac{z^{1/3}\log z}{z^2+1}\,dz + \int_{C_R} \frac{z^{1/3}\log z}{z^2+1}\,dz \\
&= \int_\rho^R \frac{r^{1/3}\ln r}{r^2+1}\,dr + \int_\rho^R \frac{r^{1/3}e^{i\pi/3}(\ln r + i\pi)}{r^2+1}\,dr + \int_{C_\rho} \frac{z^{1/3}\log z}{z^2+1}\,dz + \int_{C_R} \frac{z^{1/3}\log z}{z^2+1}\,dz \\
&= \int_\rho^R \frac{r^{1/3}\ln r(1+e^{i\pi/3})}{r^2+1}\,dr + i\pi e^{i\pi/3}\int_\rho^R \frac{r^{1/3}}{r^2+1}\,dr + \int_{C_\rho} \frac{z^{1/3}\log z}{z^2+1}\,dz + \int_{C_R} \frac{z^{1/3}\log z}{z^2+1}\,dz \\
&= (1+e^{i\pi/3})\int_\rho^R \frac{r^{1/3}\ln r}{r^2+1}\,dr + i\pi e^{i\pi/3}\int_\rho^R \frac{r^{1/3}}{r^2+1}\,dr + \int_{C_\rho} \frac{z^{1/3}\log z}{z^2+1}\,dz + \int_{C_R} \frac{z^{1/3}\log z}{z^2+1}\,dz.
\end{aligned}$$

Take the limit now as $\rho \to 0$ and $R \to \infty$. The integral over C_ρ tends to zero, and the integral over C_R tends to zero. Proof for these statements will be given at the end.

$$(1+e^{i\pi/3})\int_0^\infty \frac{r^{1/3}\ln r}{r^2+1}\,dr + i\pi e^{i\pi/3}\int_0^\infty \frac{r^{1/3}}{r^2+1}\,dr = 2\pi i \operatorname*{Res}_{z=i} \frac{z^{1/3}\log z}{z^2+1}$$

The denominator can be written as $z^2 + 1 = (z + i)(z - i)$. From this we see that the multiplicity of the factor $z - i$ is 1. The residue at $z = i$ can then be calculated by

$$\operatorname*{Res}_{z=i} \frac{z^{1/3}\log z}{z^2+1} = \phi(i),$$

where $\phi(z)$ is the same function as $f(z)$ without $(z-i)$.

$$\phi(z) = \frac{z^{1/3}\log z}{z+i} \quad \Rightarrow \quad \phi(i) = \frac{i^{1/3}\log i}{2i} = \frac{(e^{i\pi/2})^{1/3}\left(\ln 1 + i\frac{\pi}{2}\right)}{2i} = \frac{\pi}{4}e^{i\pi/6}$$

So then

$$\operatorname*{Res}_{z=i} \frac{z^{1/3}\log z}{z^2+1} = \frac{\pi}{4}e^{i\pi/6}.$$

and

$$\begin{aligned}(1+e^{i\pi/3})\int_0^\infty \frac{r^{1/3}\ln r}{r^2+1}\,dr + i\pi e^{i\pi/3}\int_0^\infty \frac{r^{1/3}}{r^2+1}\,dr &= 2\pi i\left(\frac{\pi}{4}e^{i\pi/6}\right)\\ &= i\frac{\pi^2}{2}e^{i\pi/6}.\end{aligned}$$

Use Euler's formula to separate the real and imaginary parts of the exponential functions.

$$\left(1+\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\int_0^\infty \frac{r^{1/3}\ln r}{r^2+1}\,dr + i\pi\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\int_0^\infty \frac{r^{1/3}}{r^2+1}\,dr = i\frac{\pi^2}{2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$$

$$\left(1+\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\int_0^\infty \frac{r^{1/3}\ln r}{r^2+1}\,dr + i\pi\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\int_0^\infty \frac{r^{1/3}}{r^2+1}\,dr = i\frac{\pi^2}{2}\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)$$

$$\frac{3}{2}\int_0^\infty \frac{r^{1/3}\ln r}{r^2+1}\,dr - \frac{\pi\sqrt{3}}{2}\int_0^\infty \frac{r^{1/3}}{r^2+1}\,dr + i\left(\frac{\sqrt{3}}{2}\int_0^\infty \frac{r^{1/3}\ln r}{r^2+1}\,dr + \frac{\pi}{2}\int_0^\infty \frac{r^{1/3}}{r^2+1}\,dr\right) = -\frac{\pi^2}{4} + i\frac{\pi^2\sqrt{3}}{4}$$

Match the real and imaginary parts of both sides to obtain a system of two equations.

$$\frac{3}{2}\int_0^\infty \frac{r^{1/3}\ln r}{r^2+1}\,dr - \frac{\pi\sqrt{3}}{2}\int_0^\infty \frac{r^{1/3}}{r^2+1}\,dr = -\frac{\pi^2}{4} \tag{1}$$

$$\frac{\sqrt{3}}{2}\int_0^\infty \frac{r^{1/3}\ln r}{r^2+1}\,dr + \frac{\pi}{2}\int_0^\infty \frac{r^{1/3}}{r^2+1}\,dr = \frac{\pi^2\sqrt{3}}{4} \tag{2}$$

Multiply both sides of equation (2) by $\sqrt{3}$ and then add the two equations to eliminate the integral without $\ln r$.

$$\frac{3}{2}\int_0^\infty \frac{r^{1/3}\ln r}{r^2+1}\,dr + \frac{3}{2}\int_0^\infty \frac{r^{1/3}\ln r}{r^2+1}\,dr = -\frac{\pi^2}{4} + \frac{3\pi^2}{4}$$

$$3\int_0^\infty \frac{r^{1/3}\ln r}{r^2+1}\,dr = \frac{\pi^2}{2}$$

Therefore, changing the dummy integration variable to x,

$$\boxed{\int_0^\infty \frac{\sqrt[3]{x}\ln x}{x^2+1}\,dx = \frac{\pi^2}{6}.}$$

Substitute this result into equation (1) and solve for the other integral.

$$\frac{3}{2}\left(\frac{\pi^2}{6}\right) - \frac{\pi\sqrt{3}}{2}\int_0^\infty \frac{r^{1/3}}{r^2+1}\,dr = -\frac{\pi^2}{4} \quad \rightarrow \quad -\frac{\pi\sqrt{3}}{2}\int_0^\infty \frac{r^{1/3}}{r^2+1}\,dr = -\frac{\pi^2}{2} \quad \rightarrow \quad \boxed{\int_0^\infty \frac{\sqrt[3]{x}}{x^2+1}\,dx = \frac{\pi}{\sqrt{3}}}$$

The Integral Over C_ρ

Our aim here is to show that the integral over C_ρ tends to zero in the limit as $\rho \to 0$. The parameterization of the small semicircular arc in Fig. 101 is $z = \rho e^{i\theta}$, where θ goes from π to 0.

$$\begin{aligned}
\int_{C_\rho} \frac{z^{1/3}\log z}{z^2+1}\,dz &= \int_\pi^0 \frac{(\rho e^{i\theta})^{1/3}\log(\rho e^{i\theta})}{(\rho e^{i\theta})^2+1}(\rho i e^{i\theta}\,d\theta) \\
&= \int_\pi^0 \frac{\rho^{1/3}e^{i\theta/3}(\ln\rho + i\theta)}{\rho^2 e^{2i\theta}+1}(\rho i e^{i\theta}\,d\theta) \\
&= \int_\pi^0 \frac{\rho^{4/3}\ln\rho\left(1+\frac{i\theta}{\ln\rho}\right)}{\rho^2 e^{2i\theta}+1}(ie^{4i\theta/3}\,d\theta)
\end{aligned}$$

In the limit as $\rho \to 0$, we have

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{z^{1/3}\log z}{z^2+1}\,dz = \lim_{\rho\to 0}\int_\pi^0 \frac{\rho^{4/3}\ln\rho\left(1+\frac{i\theta}{\ln\rho}\right)}{\rho^2 e^{2i\theta}+1}(ie^{4i\theta/3}\,d\theta).$$

Because the limits of integration are constant, the limit may be brought inside the integral.

$$\begin{aligned}
\lim_{\rho\to 0}\int_{C_\rho} \frac{z^{1/3}\log z}{z^2+1}\,dz &= \int_\pi^0 \lim_{\rho\to 0}\frac{\rho^{4/3}\ln\rho\left(1+\frac{i\theta}{\ln\rho}\right)}{\rho^2 e^{2i\theta}+1}(ie^{4i\theta/3}\,d\theta) \\
&= \int_\pi^0 \left[\lim_{\rho\to 0}\rho^{4/3}\ln\rho\right]\left[\lim_{\rho\to 0}\frac{1+\frac{i\theta}{\ln\rho}}{\rho^2 e^{2i\theta}+1}\right](ie^{4i\theta/3}\,d\theta) \\
&= \int_\pi^0 \left[\lim_{\rho\to 0}\frac{\ln\rho}{\rho^{-4/3}}\right][1](ie^{4i\theta/3}\,d\theta)
\end{aligned}$$

Plugging $\rho = 0$ in the remaining limit results in the indeterminate form ∞/∞, so l'Hôpital's rule will be applied to calculate it.

$$\begin{aligned}
&\overset{\frac{\infty}{\infty}}{\underset{\text{H}}{=}} \int_\pi^0 \left[\lim_{\rho\to 0}\frac{\frac{1}{\rho}}{-\frac{4}{3}\rho^{-7/3}}\right](ie^{4i\theta/3}\,d\theta) \\
&= \int_\pi^0 \left[-\frac{3}{4}\lim_{\rho\to 0}\rho^{4/3}\right](ie^{4i\theta/3}\,d\theta) \\
&= 0
\end{aligned}$$

Therefore,

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{z^{1/3}\log z}{z^2+1}\,dz = 0.$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \to \infty$. The parameterization of the large semicircular arc in Fig. 101 is $z = Re^{i\theta}$, where θ goes from 0 to π.

$$\begin{aligned}\int_{C_R} \frac{z^{1/3}\log z}{z^2+1}\,dz &= \int_0^\pi \frac{(Re^{i\theta})^{1/3}\log(Re^{i\theta})}{(Re^{i\theta})^2+1}(Rie^{i\theta}\,d\theta) \\ &= \int_0^\pi \frac{R^{4/3}e^{4i\theta/3}(\ln R + i\theta)}{R^2e^{2i\theta}+1}(i\,d\theta)\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}\left|\int_{C_R} \frac{z^{1/3}\log z}{z^2+1}\,dz\right| &= \left|\int_0^\pi \frac{R^{4/3}e^{4i\theta/3}(\ln R + i\theta)}{R^2e^{2i\theta}+1}(i\,d\theta)\right| \\ &\leq \int_0^\pi \left|\frac{R^{4/3}e^{4i\theta/3}(\ln R + i\theta)}{R^2e^{2i\theta}+1}(i)\right|d\theta \\ &= \int_0^\pi \frac{\left|R^{4/3}e^{4i\theta/3}(\ln R + i\theta)\right|}{|R^2e^{2i\theta}+1|}\,|i|\,d\theta \\ &= \int_0^\pi \frac{R^{4/3}\,|\ln R + i\theta|}{|R^2e^{2i\theta}+1|}\,d\theta \\ &\leq \int_0^\pi \frac{R^{4/3}(|\ln R| + |i\theta|)}{|R^2e^{2i\theta}| - |1|}\,d\theta \\ &= \int_0^\pi \frac{R^{4/3}(\ln R + \theta)}{R^2-1}\,d\theta \\ &= \int_0^\pi \frac{R^{4/3}\ln R\left(1+\frac{\theta}{\ln R}\right)}{R^2\left(1-\frac{1}{R^2}\right)}\,d\theta\end{aligned}$$

So we have

$$\left|\int_{C_R} \frac{z^{1/3}\log z}{z^2+1}\,dz\right| \leq \int_0^\pi \frac{\ln R\left(1+\frac{\theta}{\ln R}\right)}{R^{2/3}\left(1-\frac{1}{R^2}\right)}\,d\theta.$$

Take the limit of both sides as $R \to \infty$.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{z^{1/3}\log z}{z^2+1}\,dz\right| \leq \lim_{R\to\infty}\int_0^\pi \frac{\ln R\left(1+\frac{\theta}{\ln R}\right)}{R^{2/3}\left(1-\frac{1}{R^2}\right)}\,d\theta$$

Because the limits of integration are constant, the limit may be brought inside the integral.

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{C_R} \frac{z^{1/3}\log z}{z^2+1}\,dz\right| &\leq \int_0^\pi \lim_{R\to\infty}\frac{\ln R\left(1+\frac{\theta}{\ln R}\right)}{R^{2/3}\left(1-\frac{1}{R^2}\right)}\,d\theta \\ &= \int_0^\pi \left[\lim_{R\to\infty}\frac{\ln R}{R^{2/3}}\right]\left[\lim_{R\to\infty}\frac{1+\frac{\theta}{\ln R}}{1-\frac{1}{R^2}}\right]d\theta\end{aligned}$$

$$\lim_{R\to\infty}\left|\int_{C_R}\frac{z^{1/3}\log z}{z^2+1}\,dz\right| \leq \int_0^\pi \left[\lim_{R\to\infty}\frac{\ln R}{R^{2/3}}\right][1]\,d\theta$$

The remaining limit is the indeterminate form ∞/∞, so l'Hôpital's rule will be applied to calculate it.

$$\begin{aligned}
&\overset{\frac{\infty}{\infty}}{\underset{\text{H}}{=}} \int_0^\pi \left[\lim_{R\to\infty}\frac{\frac{1}{R}}{\frac{2}{3}R^{-1/3}}\right]d\theta \\
&= \int_0^\pi \left[\frac{3}{2}\lim_{R\to\infty}\frac{1}{R^{2/3}}\right]d\theta \\
&= 0
\end{aligned}$$

So we have

$$\lim_{R\to\infty}\left|\int_{C_R}\frac{z^{1/3}\log z}{z^2+1}\,dz\right| \leq 0.$$

The magnitude of a number cannot be negative.

$$\lim_{R\to\infty}\left|\int_{C_R}\frac{z^{1/3}\log z}{z^2+1}\,dz\right| = 0$$

The only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R\to\infty}\int_{C_R}\frac{z^{1/3}\log z}{z^2+1}\,dz = 0.$$