Exercise 8

Consider the two simple closed contours shown in Fig. 104 and obtained by dividing into two pieces the annulus formed by the circles $C_\rho$ and $C_R$ in Fig. 103 (Sec. 84). The legs $L$ and $-L$ of those contours are directed line segments along any ray $\arg z = \theta_0$, where $\pi < \theta_0 < 3\pi/2$. Also, $\Gamma_\rho$ and $\gamma_\rho$ are the indicated portions of $C_\rho$, while $\Gamma_R$ and $\gamma_R$ make up $C_R$.

(a) Show how it follows from Cauchy’s residue theorem that when the branch

$$f_1(z) = \frac{z^{-a}}{z+1} \quad (|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2})$$

of the multiple-valued function $z^{-a}/(z + 1)$ is integrated around the closed contour on the left in Fig. 104,

$$\int_{\Gamma_R} f_1(z) \, dz + \int_{\Gamma_\rho} f_1(z) \, dz + \int_{L} f_1(z) \, dz + \int_{\gamma_\rho} f_1(z) \, dz = 2\pi i \text{ Res}_{z=-1} f_1(z).$$

(b) Apply the Cauchy-Goursat theorem to the branch

$$f_2(z) = \frac{z^{-a}}{z+1} \quad (|z| > 0, \frac{\pi}{2} < \arg z < \frac{5\pi}{2})$$

of $z^{-a}/(z + 1)$, integrated around the closed contour on the right in Fig. 104, to show that

$$-\int_{\Gamma_R} f_2(z) \, dz + \int_{\Gamma_\rho} f_2(z) \, dz - \int_{L} f_2(z) \, dz + \int_{\gamma_R} f_2(z) \, dz = 0.$$

(c) Point out why, in the last lines in parts (a) and (b), the branches $f_1(z)$ and $f_2(z)$ of $z^{-a}/(z + 1)$ can be replaced by the branch

$$f(z) = \frac{z^{-a}}{z+1} \quad (|z| > 0, 0 < \arg z < 2\pi).$$

Then, by adding corresponding sides of those two lines, derive equation (3), Sec. 84, which was obtained only formally there.