Exercise 8

Consider the two simple closed contours shown in Fig. 104 and obtained by dividing into two pieces the annulus formed by the circles C_ρ and C_R in Fig. 103 (Sec. 84). The legs L and $-L$ of those contours are directed line segments along any ray $\arg z = \theta_0$, where $\pi < \theta_0 < 3\pi/2$. Also, Γ_ρ and γ_ρ are the indicated portions of C_ρ, while Γ_R and γ_R make up C_R.

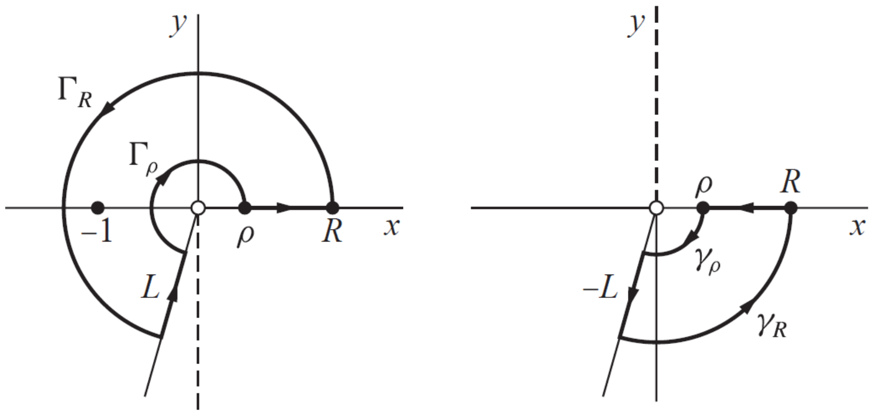

FIGURE 104

(a) Show how it follows from Cauchy's residue theorem that when the branch

$$f_1(z) = \frac{z^{-a}}{z+1} \qquad \left(|z| > 0,\ -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

of the multiple-valued function $z^{-a}/(z+1)$ is integrated around the closed contour on the left in Fig. 104,

$$\int_\rho^R \frac{r^{-a}}{r+1}\,dr + \int_{\Gamma_R} f_1(z)\,dz + \int_L f_1(z)\,dz + \int_{\Gamma_\rho} f_1(z)\,dz = 2\pi i \operatorname*{Res}_{z=-1} f_1(z).$$

(b) Apply the Cauchy-Goursat theorem to the branch

$$f_2(z) = \frac{z^{-a}}{z+1} \qquad \left(|z| > 0,\ \frac{\pi}{2} < \arg z < \frac{5\pi}{2}\right)$$

of $z^{-a}/(z+1)$, integrated around the closed contour on the right in Fig. 104, to show that

$$-\int_\rho^R \frac{r^{-a}e^{-i2a\pi}}{r+1}\,dr + \int_{\gamma_\rho} f_2(z)\,dz - \int_L f_2(z)\,dz + \int_{\gamma_R} f_2(z)\,dz = 0.$$

(c) Point out why, in the last lines in parts (a) and (b), the branches $f_1(z)$ and $f_2(z)$ of $z^{-a}/(z+1)$ can be replaced by the branch

$$f(z) = \frac{z^{-a}}{z+1} \qquad (|z| > 0,\ 0 < \arg z < 2\pi).$$

Then, by adding corresponding sides of those two lines, derive equation (3), Sec. 84, which was obtained only formally there.