Exercise 1

Use residues to evaluate the definite integrals in Exercises 1 through 7.

\[ \int_0^{2\pi} \frac{d\theta}{5 + 4\sin \theta}. \]

**Ans.** \( \frac{2\pi}{3} \).

**Solution**

Because the integral goes from 0 to \( 2\pi \), it can be thought of as one over the unit circle in the complex plane.

This circle is parameterized in terms of \( \theta \) by \( z = e^{i\theta} = \cos \theta + i \sin \theta \). Solve for \( \sin \theta \) and \( d\theta \) in terms of \( z \) and \( dz \), respectively.

\[
\begin{align*}
  z = e^{i\theta} &= \cos \theta + i \sin \theta \\
  z^{-1} = e^{-i\theta} &= \cos \theta - i \sin \theta \\
  \rightarrow z - z^{-1} &= 2i \sin \theta \\
  \rightarrow \sin \theta &= \frac{z - z^{-1}}{2i}
\end{align*}
\]

\[
\begin{align*}
  z &= e^{i\theta} \rightarrow dz = ie^{i\theta} d\theta = iz d\theta \\
  \rightarrow d\theta &= \frac{dz}{iz}
\end{align*}
\]

With this change of variables the integral in \( d\theta \) will become a positively oriented closed loop integral over the circle’s boundary \( C \).

\[
\int_0^{2\pi} \frac{d\theta}{5 + 4\sin \theta} = \oint_C \frac{1}{5 + 4 \left( \frac{z - z^{-1}}{2i} \right)} \frac{dz}{iz} = \oint_C \frac{dz}{5iz + 2z(z - z^{-1})}
\]

www.stemjock.com
\[
\int_0^{2\pi} \frac{d\theta}{5 + 4 \sin \theta} = \oint_C \frac{dz}{2z^2 + 5iz - 2} = \oint_C \frac{dz}{(z + 2i)(2z + i)} = \oint_C \frac{dz}{2(z + 2i)(z + \frac{i}{2})}
\]

According to the Cauchy residue theorem, such an integral in the complex plane is equal to \(2\pi i\) times the sum of the residues inside \(C\). Because there is only one singular point inside the unit circle, namely \(z = -i/2\), there is only one residue to calculate.

\[
\oint_C \frac{dz}{2(z + 2i)(z + \frac{i}{2})} = 2\pi i \frac{1}{\text{Res}_{z=-\frac{i}{2}} \frac{1}{2(z + 2i)(z + \frac{i}{2})}}
\]

The multiplicity of the factor \(z + i/2\) is 1, so the residue is calculated by

\[
\text{Res}_{z=-\frac{i}{2}} \frac{1}{2(z + 2i)(z + \frac{i}{2})} = \phi \left( -\frac{i}{2} \right),
\]

where \(\phi(z)\) is the same function as the integrand without the factor \(z + i/2\).

\[
\phi(z) = \frac{1}{2(z + 2i)}
\]

So then

\[
\text{Res}_{z=-\frac{i}{2}} \frac{1}{2(z + 2i)(z + \frac{i}{2})} = \frac{1}{2(-\frac{i}{2} + 2i)} = \frac{1}{3i}
\]

and

\[
\oint_C \frac{dz}{2(z + 2i)(z + \frac{i}{2})} = 2\pi i \left( \frac{1}{3i} \right) = \frac{2\pi}{3}.
\]

Therefore,

\[
\int_0^{2\pi} \frac{d\theta}{5 + 4 \sin \theta} = \frac{2\pi}{3}.
\]