Exercise 2

Use residues to evaluate the definite integrals in Exercises 1 through 7.

\[ \int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2 \theta}. \]

Ans. \( \sqrt{2\pi} \).

Solution

Start off by making the substitution,

\[ \alpha = \theta + \pi \quad \rightarrow \quad \theta = \alpha - \pi \]

\[ d\alpha = d\theta, \]

so that the integral goes from 0 to \( 2\pi \).

\[ \int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2 \theta} = \int_{-\pi + \pi}^{\pi + \pi} \frac{d\alpha}{1 + \sin^2 (\alpha - \pi)} = \int_{0}^{2\pi} \frac{d\alpha}{1 + \sin^2 \alpha} \]

The integral can now be thought of as one over the unit circle in the complex plane.

![Figure 1: This figure illustrates the unit circle in the complex plane, where \( z = x + iy \).](image)

This circle is parameterized in terms of \( \alpha \) by \( z = e^{i\alpha} = \cos \alpha + i \sin \alpha \). Solve for \( \sin \alpha \) and \( d\alpha \) in terms of \( z \) and \( dz \), respectively.

\[ \begin{cases} z = e^{i\alpha} = \cos \alpha + i \sin \alpha \\ z^{-1} = e^{-i\alpha} = \cos \alpha - i \sin \alpha \end{cases} \quad \rightarrow \quad z - z^{-1} = 2i \sin \alpha \quad \rightarrow \quad \sin \alpha = \frac{z - z^{-1}}{2i} \]

\[ z = e^{i\alpha} \quad \rightarrow \quad dz = ie^{i\alpha} d\alpha = iz \, d\alpha \quad \rightarrow \quad d\alpha = \frac{dz}{iz} \]

With this change of variables the integral in \( d\alpha \) will become a positively oriented closed loop.

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integral over the circle’s boundary $C$.

\[
\int_0^{2\pi} \frac{d\alpha}{1 + \sin^2 \alpha} = \oint_C \frac{1}{1 + \left(\frac{z - z_1}{z - z_2}\right)^2} \frac{dz}{iz}
\]

\[
= \oint_C \frac{1}{\frac{3}{2} - \frac{1}{4iz} - \frac{z^2}{4}} \cdot 4iz \cdot dz
\]

\[
= \oint_C \frac{4iz}{z^4 - 6z^2 + 1} dz
\]

According to the Cauchy residue theorem, such an integral in the complex plane is equal to $2\pi i$ times the sum of the residues inside $C$. Determine the four singular points of the integrand by solving for the roots of the denominator.

\[
z^4 - 6z^2 + 1 = 0
\]

\[
z^2 = \frac{6 \pm \sqrt{36 - 4}}{2} = 3 \pm 2\sqrt{2} \quad \rightarrow \quad \begin{cases}
z_1 = -\sqrt{3 + 2\sqrt{2}} = -1 - \sqrt{2} \approx -2.414 \\
z_2 = -\sqrt{3 - 2\sqrt{2}} = 1 - \sqrt{2} \approx -0.414 \\
z_3 = \sqrt{3 - 2\sqrt{2}} = -1 + \sqrt{2} \approx 0.414 \\
z_4 = \sqrt{3 + 2\sqrt{2}} = 1 + \sqrt{2} \approx 2.414
\end{cases}
\]

Because there are only two singular points inside the unit circle, namely $z = z_2$ and $z = z_3$, there are only two residues to calculate.

\[
\oint_C \frac{4iz}{z^4 - 6z^2 + 1} dz = 2\pi i \left( \text{Res}_{z=z_2} \frac{4iz}{z^4 - 6z^2 + 1} + \text{Res}_{z=z_3} \frac{4iz}{z^4 - 6z^2 + 1} \right)
\]

The denominator can be factored as $z^4 - 6z^2 + 1 = (z - z_1)(z - z_2)(z - z_3)(z - z_4)$. From this we see that the multiplicities of $z - z_2$ and $z - z_3$ are both 1, so the residues are calculated by

\[
\text{Res}_{z=z_2} \frac{4iz}{z^4 - 6z^2 + 1} = \phi_2(z_2)
\]

\[
\text{Res}_{z=z_3} \frac{4iz}{z^4 - 6z^2 + 1} = \phi_3(z_3),
\]

where $\phi_2(z)$ and $\phi_3(z)$ are the same function as the integrand without the factors $z - z_2$ and $z - z_3$, respectively.

\[
\phi_2(z) = \frac{4iz}{(z - z_1)(z - z_3)(z - z_4)}
\]

\[
\phi_3(z) = \frac{4iz}{(z - z_1)(z - z_2)(z - z_4)}
\]

So then

\[
\text{Res}_{z=z_2} \frac{4iz}{z^4 - 6z^2 + 1} = \frac{4iz_2}{(z_2 - z_1)(z_2 - z_3)(z_2 - z_4)} = \frac{4i(1 - \sqrt{2})}{2(2 - 2\sqrt{2})(-2\sqrt{2})} = \frac{-i\sqrt{2}}{4}
\]

\[
\text{Res}_{z=z_3} \frac{4iz}{z^4 - 6z^2 + 1} = \frac{4iz_3}{(z_3 - z_1)(z_3 - z_2)(z_3 - z_4)} = \frac{4i(-1 + \sqrt{2})}{(2\sqrt{2})(-2 + 2\sqrt{2})(-2)} = \frac{-i\sqrt{2}}{4}
\]

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and
\[ \oint_C \frac{4iz \, dz}{z^4 - 6z^2 + 1} = 2\pi i \left( -\frac{i\sqrt{2}}{4} - \frac{i\sqrt{2}}{4} \right) = \sqrt{2}\pi \]

and
\[ \int_0^{2\pi} \frac{d\alpha}{1 + \sin^2 \alpha} = \sqrt{2}\pi. \]

Therefore,
\[ \int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2 \theta} = \sqrt{2}\pi. \]