Exercise 3

Use residues to evaluate the definite integrals in Exercises 1 through 7.

$$\int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{5 - 4\cos 2\theta}.$$

$$Ans.\ \frac{3\pi}{8}.$$

Solution

Because the integral goes from 0 to 2π, it can be thought of as one over the unit circle in the complex plane.

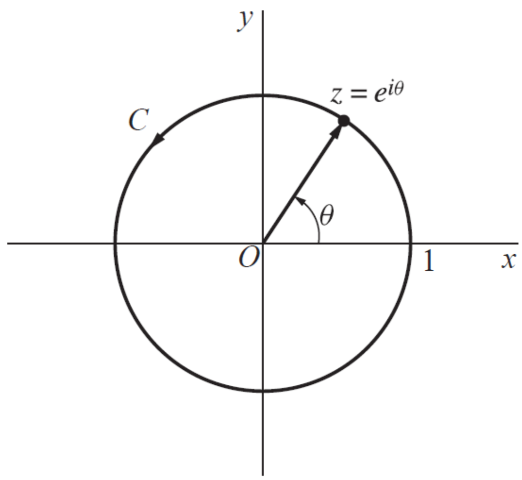

Figure 1: This figure illustrates the unit circle in the complex plane, where $z = x + iy$.

This circle is parameterized in terms of θ by $z = e^{i\theta} = \cos\theta + i\sin\theta$. Write $\cos 3\theta$ and $\cos 2\theta$ in terms of z and write $d\theta$ in terms of dz.

$$\begin{cases} z^3 = e^{3i\theta} = \cos 3\theta + i\sin 3\theta \\ z^{-3} = e^{-3i\theta} = \cos 3\theta - i\sin 3\theta \end{cases} \quad \to \quad z^3 + z^{-3} = 2\cos 3\theta \quad \to \quad \cos 3\theta = \frac{z^3 + z^{-3}}{2}$$

$$\begin{cases} z^2 = e^{2i\theta} = \cos 2\theta + i\sin 2\theta \\ z^{-2} = e^{-2i\theta} = \cos 2\theta - i\sin 2\theta \end{cases} \quad \to \quad z^2 + z^{-2} = 2\cos 2\theta \quad \to \quad \cos 2\theta = \frac{z^2 + z^{-2}}{2}$$

$$z = e^{i\theta} \quad \to \quad dz = ie^{i\theta}\, d\theta = iz\, d\theta \quad \to \quad d\theta = \frac{dz}{iz}$$

With this change of variables the integral in $d\theta$ will become a positively oriented closed loop integral over the circle's boundary C.

$$\begin{aligned} \int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{5 - 4\cos 2\theta} &= \oint_C \frac{\left(\frac{z^3 + z^{-3}}{2}\right)^2}{5 - 4\left(\frac{z^2 + z^{-2}}{2}\right)} \frac{dz}{iz} \\ &= \oint_C \frac{z^6 + 2 + z^{-6}}{5 - 2z^2 - 2z^{-2}} \frac{dz}{4iz} \end{aligned}$$

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos^2 3\theta\, d\theta}{5 - 4\cos 2\theta} &= \oint_C \frac{z^{12} + 2z^6 + 1}{5 - 2z^2 - 2z^{-2}} \frac{dz}{4iz^7} \\
&= \oint_C \frac{z^{12} + 2z^6 + 1}{2z^4 - 5z^2 + 2} \frac{i\, dz}{4z^5} \\
&= \oint_C \frac{i(z^6+1)^2}{8z^5\left(z^4 - \frac{5}{2}z^2 + 1\right)}\, dz
\end{aligned}$$

According to the Cauchy residue theorem, such an integral in the complex plane is equal to $2\pi i$ times the sum of the residues inside C. Determine the singular points of the integrand by solving for the roots of the denominator.

$$8z^5\left(z^4 - \frac{5}{2}z^2 + 1\right) = 0$$

$$z = 0 \quad \text{or} \quad z^4 - \frac{5}{2}z^2 + 1 = 0$$

$$z^2 = \frac{\frac{5}{2} \pm \sqrt{\frac{25}{4} - 4}}{2} = \frac{5}{4} \pm \frac{3}{4} \quad \rightarrow \quad \begin{cases} z_1 = -\sqrt{2} \approx -1.414 \\ z_2 = -\dfrac{1}{\sqrt{2}} \approx -0.707 \\ z_3 = \dfrac{1}{\sqrt{2}} \approx 0.707 \\ z_4 = \sqrt{2} \approx 1.414 \end{cases}$$

Because there are only three singular points inside the unit circle, namely $z = 0$ and $z = z_2$ and $z = z_3$, there are only three residues to calculate.

$$\oint_C \frac{i(z^6+1)^2}{8z^5\left(z^4 - \frac{5}{2}z^2 + 1\right)}\, dz = 2\pi i \left[\operatorname*{Res}_{z=0} \frac{i(z^6+1)^2}{8z^5\left(z^4 - \frac{5}{2}z^2 + 1\right)} + \operatorname*{Res}_{z=z_2} \frac{i(z^6+1)^2}{8z^5\left(z^4 - \frac{5}{2}z^2 + 1\right)} + \operatorname*{Res}_{z=z_3} \frac{i(z^6+1)^2}{8z^5\left(z^4 - \frac{5}{2}z^2 + 1\right)}\right]$$

The denominator can be factored as $8z^5\left(z^4 - \frac{5}{2}z^2 + 1\right) = 8z^5(z - z_1)(z - z_2)(z - z_3)(z - z_4)$. From this we see that the multiplicities of $z - z_2$ and $z - z_3$ are both 1 and that the multiplicity of z is 5, so the residues are calculated by

$$\begin{aligned}
\operatorname*{Res}_{z=0} \frac{i(z^6+1)^2}{8z^5\left(z^4 - \frac{5}{2}z^2 + 1\right)} &= \frac{\phi_0^{(5-1)}(0)}{(5-1)!} = \frac{\phi_0^{(4)}(0)}{24} \\
\operatorname*{Res}_{z=z_2} \frac{i(z^6+1)^2}{8z^5\left(z^4 - \frac{5}{2}z^2 + 1\right)} &= \phi_2(z_2) \\
\operatorname*{Res}_{z=z_3} \frac{i(z^6+1)^2}{8z^5\left(z^4 - \frac{5}{2}z^2 + 1\right)} &= \phi_3(z_3),
\end{aligned}$$

where $\phi_0(z)$ and $\phi_2(z)$ and $\phi_3(z)$ are the same function as the integrand without the factors z^5 and $z - z_2$ and $z - z_3$, respectively.

$$\begin{aligned}
\phi_0(z) &= \frac{i(z^6+1)^2}{8\left(z^4 - \frac{5}{2}z^2 + 1\right)} \\
\phi_2(z) &= \frac{i(z^6+1)^2}{8z^5(z - z_1)(z - z_3)(z - z_4)} \\
\phi_3(z) &= \frac{i(z^6+1)^2}{8z^5(z - z_1)(z - z_2)(z - z_4)}
\end{aligned}$$

Rather than taking four derivatives of $\phi_0(z)$, an alternative approach to finding the residue at $z = 0$ would be to use long division to divide $i(z^6+1)^2$ by $8z^5\left(z^4 - \frac{5}{2}z^2 + 1\right)$. The residue would then just be the coefficient of $1/z$.

$$\operatorname*{Res}_{z=0} \frac{i(z^6+1)^2}{8z^5\left(z^4 - \frac{5}{2}z^2 + 1\right)} = \frac{\phi_0^{(4)}(0)}{24} = \frac{21i}{32}$$
$$\operatorname*{Res}_{z=z_2} \frac{i(z^6+1)^2}{8z^5\left(z^4 - \frac{5}{2}z^2 + 1\right)} = \frac{i(z_2^6+1)^2}{8z_2^5(z_2-z_1)(z_2-z_3)(z_2-z_4)} = -\frac{27}{64}i$$
$$\operatorname*{Res}_{z=z_3} \frac{i(z^6+1)^2}{8z^5\left(z^4 - \frac{5}{2}z^2 + 1\right)} = \frac{i(z_3^6+1)^2}{8z_3^5(z_3-z_1)(z_3-z_2)(z_3-z_4)} = -\frac{27}{64}i$$

As a result,

$$\oint_C \frac{i(z^6+1)^2}{8z^5\left(z^4 - \frac{5}{2}z^2 + 1\right)} = 2\pi i\left(\frac{21i}{32} - \frac{27}{64}i - \frac{27}{64}i\right) = 2\pi i\left(-\frac{6i}{32}\right) = \frac{3\pi}{8}.$$

Therefore,

$$\int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{5 - 4\cos 2\theta} = \frac{3\pi}{8}.$$