Exercise 4

Use residues to evaluate the definite integrals in Exercises 1 through 7.

$$\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} \quad (-1 < a < 1).$$

$$Ans.\ \frac{2\pi}{\sqrt{1-a^2}}.$$

Solution

Because the integral goes from 0 to 2π, it can be thought of as one over the unit circle in the complex plane.

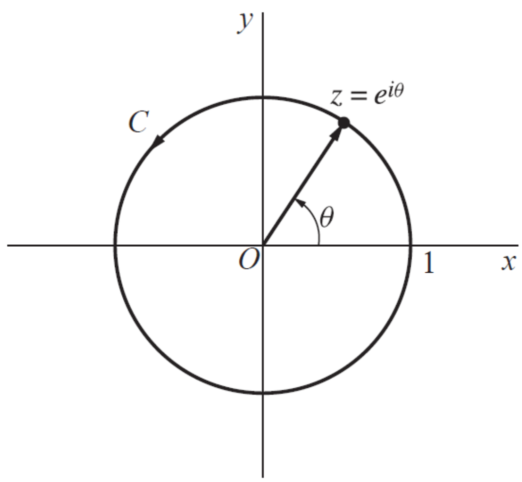

Figure 1: This figure illustrates the unit circle in the complex plane, where $z = x + iy$.

This circle is parameterized in terms of θ by $z = e^{i\theta} = \cos\theta + i\sin\theta$. Solve for $\cos\theta$ and $d\theta$ in terms of z and dz, respectively.

$$\begin{cases} z = e^{i\theta} = \cos\theta + i\sin\theta \\ z^{-1} = e^{-i\theta} = \cos\theta - i\sin\theta \end{cases} \quad \to \quad z + z^{-1} = 2\cos\theta \quad \to \quad \cos\theta = \frac{z + z^{-1}}{2}$$

$$z = e^{i\theta} \quad \to \quad dz = ie^{i\theta}\,d\theta = iz\,d\theta \quad \to \quad d\theta = \frac{dz}{iz}$$

With this change of variables the integral in $d\theta$ will become a positively oriented closed loop integral over the circle's boundary C.

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} &= \oint_C \frac{1}{1 + a\left(\frac{z+z^{-1}}{2}\right)} \frac{dz}{iz} \\ &= \oint_C \frac{2}{z^2 + \frac{2}{a}z + 1} \frac{dz}{ia} \end{aligned}$$

According to the Cauchy residue theorem, such an integral in the complex plane is equal to $2\pi i$ times the sum of the residues inside C. Determine the two singular points of the integrand by solving for the roots of the denominator.

$$ia\left(z^2 + \frac{2}{a}z + 1\right) = 0$$

$$z^2 + \frac{2}{a}z + 1 = 0$$

$$z = \frac{-\frac{2}{a} \pm \sqrt{\frac{4}{a^2} - 4}}{2} = -\frac{1}{a} \pm \frac{\sqrt{1-a^2}}{a} = \frac{-1 \pm \sqrt{1-a^2}}{a} \quad \to \quad \begin{cases} z_1 = \dfrac{-1+\sqrt{1-a^2}}{a} \\ z_2 = \dfrac{-1-\sqrt{1-a^2}}{a} \end{cases}$$

Since $-1 < a < 1$, there is only one singular point inside the unit circle, namely $z = z_1$, so there is only one residue to calculate.

$$\oint_C \frac{2}{z^2 + \frac{2}{a}z + 1} \frac{dz}{ia} = 2\pi i \operatorname*{Res}_{z=z_1} \frac{2}{z^2 + \frac{2}{a}z + 1} \frac{1}{ia}$$

The denominator can be factored as $z^2 + \frac{2}{a}z + 1 = (z - z_1)(z - z_2)$. From this we see that the multiplicity of the factor $z - z_1$ is 1, so the residue is calculated by

$$\operatorname*{Res}_{z=z_1} \frac{2}{z^2 + \frac{2}{a}z + 1} \frac{1}{ia} = \phi(z_1),$$

where $\phi(z)$ is the same function as the integrand without the factor $z - z_1$.

$$\phi(z) = \frac{2}{z - z_2} \frac{1}{ia}$$

So then

$$\operatorname*{Res}_{z=z_1} \frac{2}{z^2 + \frac{2}{a}z + 1} \frac{1}{ia} = \frac{2}{z_1 - z_2} \frac{1}{ia} = \frac{2}{\frac{2}{a}\sqrt{1-a^2}} \frac{1}{ia} = \frac{1}{i\sqrt{1-a^2}}$$

and

$$\oint_C \frac{2}{z^2 + \frac{2}{a}z + 1} \frac{dz}{ia} = 2\pi i \left(\frac{1}{i\sqrt{1-a^2}}\right) = \frac{2\pi}{\sqrt{1-a^2}}.$$

Therefore,

$$\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}} \quad (-1 < a < 1).$$