Exercise 6

Use residues to evaluate the definite integrals in Exercises 1 through 7.

\[
\int_0^\pi \frac{d\theta}{(a + \cos \theta)^2} \quad (a > 1).
\]

Ans. \( \frac{a\pi}{(\sqrt{a^2 - 1})^3} \).

Solution

Notice that the integrand is an even function of \( \theta \), so the lower limit of integration can be extended to \(-\pi\) as long as the integral is divided by 2.

\[
\frac{1}{2} \int_{-\pi}^{\pi} \frac{d\theta}{(a + \cos \theta)^2} = \frac{1}{2} \int_{-\pi}^{\pi} \frac{d\theta}{(a + \cos \theta)^2}
\]

Now make the substitution,

\[
\alpha = \theta + \pi \quad \rightarrow \quad \theta = \alpha - \pi \\
d\alpha = d\theta,
\]

so that the integral goes from 0 to 2\(\pi\).

\[
\frac{1}{2} \int_{-\pi}^{\pi} \frac{d\theta}{(a + \cos \theta)^2} = \frac{1}{2} \int_{-\pi}^{\pi} \frac{d\alpha}{[a + \cos(\alpha - \pi)]^2} = \frac{1}{2} \int_0^{2\pi} \frac{d\alpha}{(a - \cos \alpha)^2}
\]

The integral can now be thought of as one over the unit circle in the complex plane.

\[
z = e^{i\alpha} = \cos \alpha + i \sin \alpha.
\]

Solve for \( \cos \alpha \) and \( d\alpha \) in terms of \( z \) and \( dz \), respectively.

\[
\begin{align*}
z &= e^{i\alpha} = \cos \alpha + i \sin \alpha \\
z^{-1} &= e^{-i\alpha} = \cos \alpha - i \sin \alpha \\
\end{align*}
\]

\[
\begin{align*}
z &= e^{i\alpha} \quad \rightarrow \quad dz = ie^{i\alpha} d\alpha = iz \, d\alpha \\
\end{align*}
\]

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With this change of variables the integral in $d\alpha$ will become a positively oriented closed loop integral over the circle’s boundary $C$.

\[
\frac{1}{2} \int_0^{2\pi} \frac{d\alpha}{(a - \cos \alpha)^2} = \oint_C \frac{1}{2} \frac{1}{(a - \frac{z + z^{-1}}{2})^2} \frac{dz}{iz} = \oint_C \frac{1}{2} \frac{4z^2}{(z^2 - 2az + 1)^2} \frac{dz}{iz} = \oint_C \frac{-2iz}{(z^2 - 2az + 1)^2} \frac{dz}{iz}
\]

According to the Cauchy residue theorem, such an integral in the complex plane is equal to $2\pi i$ times the sum of the residues inside $C$. Determine the two singular points of the integrand by solving for the roots of the denominator.

\[
(z^2 - 2az + 1)^2 = 0
\]

\[
z^2 - 2az + 1 = 0
\]

\[
z = \frac{2a \pm \sqrt{4a^2 - 4}}{2} = a \pm \sqrt{a^2 - 1}
\]

\[
\Rightarrow \begin{cases} 
  z_1 = a + \sqrt{a^2 - 1} \\
  z_2 = a - \sqrt{a^2 - 1}
\end{cases}
\]

Since $a > 1$, there is only one singular point inside the unit circle, namely $z = z_2$, so there is only one residue to calculate.

\[
\oint_C \frac{-2iz}{(z^2 - 2az + 1)^2} \frac{dz}{iz} = 2\pi i \text{ Res}_{z=z_2} \frac{-2iz}{(z^2 - 2az + 1)^2}
\]

The denominator can be factored as $(z^2 - 2az + 1)^2 = (z - z_1)^2(z - z_2)^2$. From this we see that the multiplicity of the factor $z - z_2$ is 2, so the residue is calculated by

\[
\text{Res}_{z=z_2} \frac{-2iz}{(z^2 - 2az + 1)^2} = \frac{\phi'z_2}{(2-1)!} = \phi'(z_2),
\]

where $\phi(z)$ is the same function as the integrand without the factor $(z - z_2)^2$.

\[
\phi(z) = \frac{-2iz}{(z - z_1)^2}
\]

So then

\[
\text{Res}_{z=z_2} \frac{-2iz}{(z^2 - 2az + 1)^2} = -\frac{ia}{2 \left( \sqrt{a^2 - 1} \right)^3}
\]

and

\[
\oint_C \frac{-2iz}{(z^2 - 2az + 1)^2} \frac{dz}{iz} = 2\pi i \left[ -\frac{ia}{2 \left( \sqrt{a^2 - 1} \right)^3} \right] = \frac{a\pi}{\left( \sqrt{a^2 - 1} \right)^3}.
\]

Therefore,

\[
\int_0^\pi \frac{d\theta}{(a + \cos \theta)^2} = \frac{a\pi}{\left( \sqrt{a^2 - 1} \right)^3} \quad (a > 1).
\]