Exercise 12

Suppose that a function $F(s)$ has a pole of order m at $s = s_0$, with a Laurent series expansion

$$F(s) = \sum_{n=0}^{\infty} a_n(s-s_0)^n + \frac{b_1}{s-s_0} + \frac{b_2}{(s-s_0)^2} + \cdots + \frac{b_{m-1}}{(s-s_0)^{m-1}} + \frac{b_m}{(s-s_0)^m} \quad (b_m \neq 0)$$

in the punctured disk $0 < |s - s_0| < R_2$, and note that $(s - s_0)^m F(s)$ is represented in that domain by the power series

$$b_m + b_{m-1}(s-s_0) + \cdots + b_2(s-s_0)^{m-2} + b_1(s-s_0)^{m-1} + \sum_{n=0}^{\infty} a_n(s-s_0)^{m+n}.$$

By collecting the terms that make up the coefficient of $(s - s_0)^{m-1}$ in the product (Sec. 67) of this power series and the Taylor series expansion

$$e^{st} = e^{s_0 t}\left[1 + \frac{t}{1!}(s-s_0) + \cdots + \frac{t^{m-2}}{(m-2)!}(s-s_0)^{m-2} + \frac{t^{m-1}}{(m-1)!}(s-s_0)^{m-1} + \cdots\right]$$

of the entire function $e^{st} = e^{s_0 t}e^{(s-s_0)t}$, show that

$$\operatorname*{Res}_{s=s_0}[e^{st}F(s)] = e^{s_0 t}\left[b_1 + \frac{b_2}{1!}t + \cdots + \frac{b_{m-1}}{(m-2)!}t^{m-2} + \frac{b_m}{(m-1)!}t^{m-1}\right],$$

as stated at the beginning of Sec. 89.

Solution

Our aim is to find the residue of $e^{st}F(s)$ at $s = s_0$,

$$\operatorname*{Res}_{s=s_0}[e^{st}F(s)],$$

which is the coefficient of $1/(s - s_0)$, or $(s - s_0)^{-1}$, in the Laurent series expansion of $e^{st}F(s)$.

$$e^{st} = e^{s_0 t}e^{(s-s_0)t}$$

Multiply both sides by $(s - s_0)^m F(s)$.

$$(s-s_0)^m e^{st}F(s) = (s-s_0)^m F(s) e^{s_0 t} e^{(s-s_0)t}$$

Now substitute the power series for $(s - s_0)^m F(s)$ and the Taylor series for e^{st}.

$$\begin{aligned}(s-s_0)^m e^{st}F(s) = &\left[b_m + b_{m-1}(s-s_0) + \cdots + b_2(s-s_0)^{m-2} + b_1(s-s_0)^{m-1} + \sum_{n=0}^{\infty} a_n(s-s_0)^{m+n}\right] \\ &\times e^{s_0 t}\left[1 + \frac{t}{1!}(s-s_0) + \cdots + \frac{t^{m-2}}{(m-2)!}(s-s_0)^{m-2} + \frac{t^{m-1}}{(m-1)!}(s-s_0)^{m-1} + \cdots\right]\end{aligned}$$

Distribute each of the terms in the first factor.

$$\begin{aligned}(s-s_0)^m e^{st}F(s) = e^{s_0t}\Bigg\{ & b_m\left[1+\frac{t}{1!}(s-s_0)+\cdots+\frac{t^{m-2}}{(m-2)!}(s-s_0)^{m-2}+\underbrace{\frac{t^{m-1}}{(m-1)!}(s-s_0)^{m-1}}+\cdots\right] \\ & + b_{m-1}(s-s_0)\left[1+\frac{t}{1!}(s-s_0)+\cdots+\underbrace{\frac{t^{m-2}}{(m-2)!}(s-s_0)^{m-2}}+\frac{t^{m-1}}{(m-1)!}(s-s_0)^{m-1}+\cdots\right] \\ & + \cdots \\ & + b_2(s-s_0)^{m-2}\left[1+\underbrace{\frac{t}{1!}(s-s_0)}+\cdots+\frac{t^{m-2}}{(m-2)!}(s-s_0)^{m-2}+\frac{t^{m-1}}{(m-1)!}(s-s_0)^{m-1}+\cdots\right] \\ & + b_1(s-s_0)^{m-1}\left[\underbrace{1}+\frac{t}{1!}(s-s_0)+\cdots+\frac{t^{m-2}}{(m-2)!}(s-s_0)^{m-2}+\frac{t^{m-1}}{(m-1)!}(s-s_0)^{m-1}+\cdots\right] \\ & + \sum_{n=0}^{\infty} a_n(s-s_0)^{m+n}\left[1+\frac{t}{1!}(s-s_0)+\cdots+\frac{t^{m-2}}{(m-2)!}(s-s_0)^{m-2}+\frac{t^{m-1}}{(m-1)!}(s-s_0)^{m-1}+\cdots\right]\Bigg\}\end{aligned}$$

We don't care about most of these terms; we only care about the ones with a brace underneath them because they give us a factor of $(s-s_0)^{m-1}$ upon multiplication.

$$\begin{aligned}(s-s_0)^m e^{st}F(s) = e^{s_0t}\Bigg[& b_m\frac{t^{m-1}}{(m-1)!}(s-s_0)^{m-1}+\cdots \\ & + b_{m-1}\frac{t^{m-2}}{(m-2)!}(s-s_0)^{m-1}+\cdots \\ & + \cdots \\ & + b_2\frac{t}{1!}(s-s_0)^{m-1}+\cdots \\ & + b_1(s-s_0)^{m-1}+\cdots \\ & + \cdots\Bigg]\end{aligned}$$

Factor out $(s-s_0)^{m-1}$.

$$(s-s_0)^m e^{st}F(s) = e^{s_0t}\left[b_m\frac{t^{m-1}}{(m-1)!}+b_{m-1}\frac{t^{m-2}}{(m-2)!}+\cdots+b_2\frac{t}{1!}+b_1\right](s-s_0)^{m-1}+\cdots$$

Divide both sides by $(s-s_0)^m$.

$$e^{st}F(s) = e^{s_0t}\left[b_m\frac{t^{m-1}}{(m-1)!}+b_{m-1}\frac{t^{m-2}}{(m-2)!}+\cdots+b_2\frac{t}{1!}+b_1\right](s-s_0)^{-1}+\cdots$$

The coefficient of $(s-s_0)^{-1}$ is the residue of $e^{st}F(s)$ at $s=s_0$. Therefore,

$$\operatorname*{Res}_{s=s_0}[e^{st}F(s)] = e^{s_0t}\left[b_1+\frac{b_2}{1!}t+\cdots+\frac{b_{m-1}}{(m-2)!}t^{m-2}+\frac{b_m}{(m-1)!}t^{m-1}\right].$$