Exercise 12

Suppose that a function $F(s)$ has a pole of order m at $s = s_0$, with a Laurent series expansion

$$F(s) = \sum_{n=0}^{\infty} a_n (s - s_0)^n + \frac{b_1}{s - s_0} + \frac{b_2}{(s - s_0)^2} + \cdots + \frac{b_{m-1}}{(s - s_0)^{m-1}} + \frac{b_m}{(s - s_0)^m} \quad (b_m \neq 0)$$

in the punctured disk $0 < |s - s_0| < R_2$, and note that $(s - s_0)^m F(s)$ is represented in that domain by the power series

$$b_m + b_{m-1}(s - s_0) + \cdots + b_2(s - s_0)^{m-2} + b_1(s - s_0)^{m-1} + \sum_{n=0}^{\infty} a_n (s - s_0)^{m+n}.$$

By collecting the terms that make up the coefficient of $(s - s_0)^{m-1}$ in the product (Sec. 67) of this power series and the Taylor series expansion

$$e^{st} = e^{s_0 t} \left[1 + \frac{t}{1!}(s - s_0) + \cdots + \frac{t^{m-2}}{(m-2)!}(s - s_0)^{m-2} + \frac{t^{m-1}}{(m-1)!}(s - s_0)^{m-1} + \cdots \right]$$

of the entire function $e^{st} = e^{s_0 t} e^{(s - s_0)t}$, show that

$$\operatorname*{Res}_{s=s_0} [e^{st} F(s)] = e^{s_0 t} \left[b_1 + \frac{b_2}{1!} t + \cdots + \frac{b_{m-1}}{(m-2)!} t^{m-2} + \frac{b_m}{(m-1)!} t^{m-1} \right],$$

as stated at the beginning of Sec. 89.