Exercise 13

Let the point \( s_0 = \alpha + i\beta \) \((\beta \neq 0)\) be a pole of order \( m \) of a function \( F(s) \), which has a Laurent series representation

\[
F(s) = \sum_{n=0}^{\infty} a_n(s - s_0)^n + \frac{b_1}{s - s_0} + \frac{b_2}{(s - s_0)^2} + \cdots + \frac{b_m}{(s - s_0)^m} \quad (b_m \neq 0)
\]

in the punctured disk \( 0 < |s - s_0| < R_2 \). Also, assume that \( F(s) = F(\bar{s}) \) at points \( s \) where \( F(s) \) is analytic.

(a) With the aid of the result in Exercise 6, Sec. 56, point out how it follows that

\[
F(s) = \sum_{n=0}^{\infty} a_n(s - s_0)^n + \frac{b_1}{s - s_0} + \frac{b_2}{(s - s_0)^2} + \cdots + \frac{\bar{b}_m}{(\bar{s} - \bar{s}_0)^m} \quad (\bar{b}_m \neq 0)
\]

when \( 0 < |\bar{s} - \bar{s}_0| < R_2 \). Then replace \( \bar{s} \) by \( s \) here to obtain a Laurent series representation for \( F(s) \) in the punctured disk \( 0 < |s - s_0| < R_2 \), and conclude that \( s_0 \) is a pole of order \( m \) of \( F(s) \).

(b) Use results in Exercise 12 and part (a) to show that

\[
\text{Res}_{s=s_0}[e^{\alpha t}F(s)] + \text{Res}_{s=s_0}[e^{\alpha t}\bar{F}(s)] = 2e^{\alpha t}\text{Re}\left\{e^{i\beta t}\left[b_1 + \frac{b_2}{2!}t + \cdots + \frac{b_m}{(m-1)!}t^{m-1}\right]\right\}
\]

when \( t \) is real, as stated just before Example 1 in Sec. 89.