Exercise 14

Let $F(s)$ be the function in Exercise 13, and write the nonzero coefficient b_m there in exponential form as $b_m = r_m \exp(i\theta_m)$. Then use the main result in part (b) of Exercise 13 to show that when t is real, the sum of the residues of $e^{st}F(s)$ at $s_0 = \alpha + i\beta$ $(\beta \neq 0)$ and $\overline{s_0}$ contains a term of the type

$$\frac{2r_m}{(m-1)!}t^{m-1}e^{\alpha t}\cos(\beta t + \theta_m).$$

Note that if $\alpha > 0$, the product $t^{m-1}e^{\alpha t}$ here tends to ∞ as t tends to ∞. When the inverse Laplace transform $f(t)$ is found by summing the residues of $e^{st}F(s)$, the term displayed just above is, therefore, an *unstable* component of $f(t)$ if $\alpha > 0$; and it is said to be of *resonance* type. If $m \geq 2$ and $\alpha = 0$, the term is also of resonance type.

Solution

Let $F(s)$ be the function in Exercise 13.

$$F(s) = \sum_{n=0}^{\infty} a_n(s-s_0)^n + \frac{b_1}{s-s_0} + \frac{b_2}{(s-s_0)^2} + \cdots + \frac{b_m}{(s-s_0)^m} \quad (b_m \neq 0)$$

Write the nonzero coefficient b_m there in exponential form as $b_m = r_m \exp(i\theta_m)$.

$$F(s) = \sum_{n=0}^{\infty} a_n(s-s_0)^n + \frac{b_1}{s-s_0} + \frac{b_2}{(s-s_0)^2} + \cdots + \frac{r_m \exp(i\theta_m)}{(s-s_0)^m}$$

When t is real the sum of the residues of $e^{st}F(s)$ at s_0 and $\overline{s_0}$ is, according to part (b) of Exercise 13,

$$\operatorname*{Res}_{s=s_0}[e^{st}F(s)] + \operatorname*{Res}_{s=\overline{s_0}}[e^{st}F(s)] = 2e^{\alpha t}\,\mathrm{Re}\left\{e^{i\beta t}\left[b_1 + \frac{b_2}{1!}t + \cdots + \frac{r_m \exp(i\theta_m)}{(m-1)!}t^{m-1}\right]\right\}.$$

Distribute $e^{i\beta t}$.

$$= 2e^{\alpha t}\,\mathrm{Re}\left[b_1 e^{i\beta t} + \frac{b_2}{1!}te^{i\beta t} + \cdots + \frac{r_m e^{i\beta t}\exp(i\theta_m)}{(m-1)!}t^{m-1}\right]$$

Combine the exponential functions.

$$= 2e^{\alpha t}\,\mathrm{Re}\left[b_1 e^{i\beta t} + \frac{b_2}{1!}te^{i\beta t} + \cdots + \frac{r_m e^{i(\beta t+\theta_m)}}{(m-1)!}t^{m-1}\right]$$

Split up the real part and distribute $2e^{\alpha t}$.

$$= 2e^{\alpha t}\,\mathrm{Re}\left[b_1 e^{i\beta t}\right] + \cdots + 2e^{\alpha t}\,\mathrm{Re}\left[\frac{r_m e^{i(\beta t+\theta_m)}}{(m-1)!}t^{m-1}\right]$$

Focus now on the last term and use Euler's formula.

$$2e^{\alpha t}\operatorname{Re}\left[\frac{r_m e^{i(\beta t+\theta_m)}}{(m-1)!}t^{m-1}\right] = 2e^{\alpha t}\operatorname{Re}\left\{\frac{r_m}{(m-1)!}t^{m-1}[\cos(\beta t+\theta_m)+i\sin(\beta t+\theta_m)]\right\}$$

Expand the term in curly braces.

$$= 2e^{\alpha t}\operatorname{Re}\left[\frac{r_m}{(m-1)!}t^{m-1}\cos(\beta t+\theta_m)+i\frac{r_m}{(m-1)!}t^{m-1}\sin(\beta t+\theta_m)\right]$$

Take the real part.

$$= 2e^{\alpha t}\frac{r_m}{(m-1)!}t^{m-1}\cos(\beta t+\theta_m)$$

Therefore, when t is real, the sum of the residues of $e^{st}F(s)$ at s_0 and $\overline{s_0}$ contains a term of the type

$$\frac{2r_m}{(m-1)!}t^{m-1}e^{\alpha t}\cos(\beta t+\theta_m).$$