

## Exercise 5

In Exercises 1 through 5, use the method described in Sec. 88 and illustrated in Example 1, Sec. 89, to find the function  $f(t)$  corresponding to the given function  $F(s)$ .

$$F(s) = \frac{8a^3 s^2}{(s^2 + a^2)^3} \quad (a > 0).$$

*Suggestion:* Refer to Exercise 5, Sec. 72, for the principal part of  $F(s)$  at  $s = ai$ .

*Ans.*  $f(t) = (1 + a^2 t^2) \sin at - at \cos at$ .

### Solution

The inverse Laplace transform of the given function for  $F(s)$  is defined by the Bromwich integral,

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} \frac{8a^3 s^2}{(s^2 + a^2)^3} ds,$$

where  $\gamma$  is a real constant chosen such that all singularities of the integrand lie to the left of the infinite vertical line  $(\gamma - i\infty, \gamma + i\infty)$  in the complex plane. They occur where the denominator is equal to zero.

$$\begin{aligned} (s^2 + a^2)^3 &= 0 \\ s^2 + a^2 &= 0 \\ (s + ia)(s - ia) &= 0 \quad \rightarrow \quad \begin{cases} s_1 = -ia \\ s_2 = ia \end{cases} \end{aligned}$$

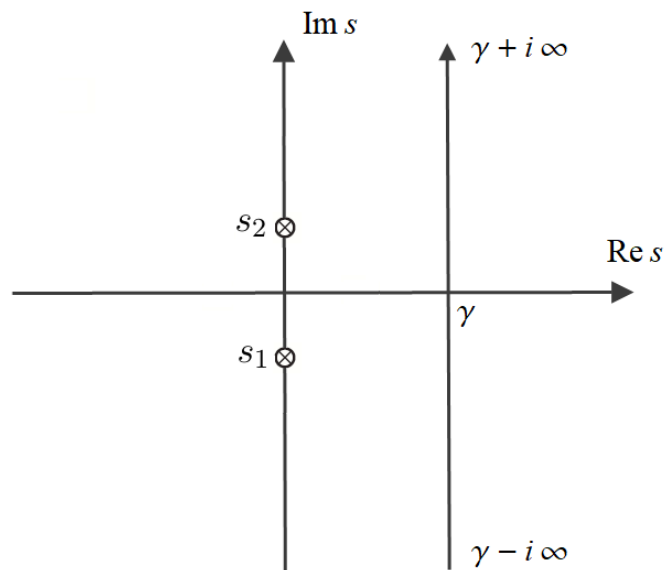


Figure 1: This is the complex plane with the singularities of the integrand marked as well as the vertical line  $(\gamma - i\infty, \gamma + i\infty)$ .

The integral is evaluated by considering a closed loop integral in the complex plane containing this vertical line and then applying the Cauchy residue theorem to get an equation, allowing us to solve for it. Let the vertical line loop around to the bottom by a semicircular arc to the left as illustrated in Figure 2 so that the integral is positively oriented.

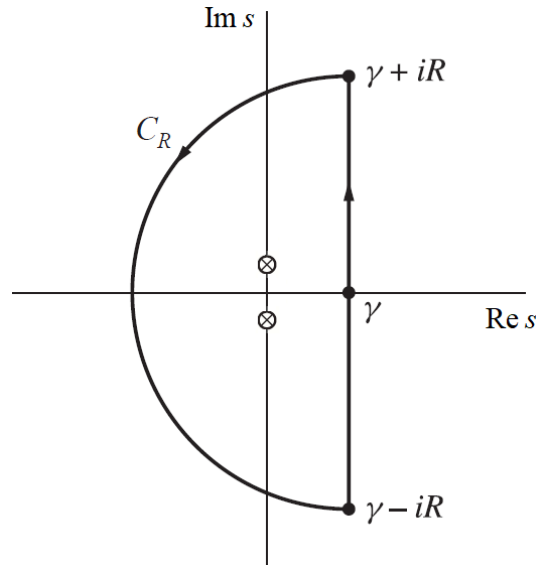


Figure 2: This is the closed loop that will be considered to calculate the inverse Laplace transform.

Now that the integration path is closed, the Cauchy residue theorem can be applied, which states that the integral over this path is equal to  $2\pi i$  times the sum of the residues inside the loop.

$$\oint_C e^{st} \frac{8a^3 s^2}{(s^2 + a^2)^3} ds = 2\pi i \sum_{n=1}^2 \operatorname{Res}_{s=s_n} e^{st} \frac{8a^3 s^2}{(s^2 + a^2)^3}$$

Expand both sides.

$$\int_{\gamma-iR}^{\gamma+iR} e^{st} \frac{8a^3 s^2}{(s^2 + a^2)^3} ds + \int_{C_R} e^{st} \frac{8a^3 s^2}{(s^2 + a^2)^3} ds = 2\pi i \left[ \operatorname{Res}_{s=s_1} e^{st} \frac{8a^3 s^2}{(s^2 + a^2)^3} + \operatorname{Res}_{s=s_2} e^{st} \frac{8a^3 s^2}{(s^2 + a^2)^3} \right]$$

In the limit as  $R \rightarrow \infty$  the integral over  $C_R$  tends to zero. Proof for this statement will be given at the end.

$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} \frac{8a^3 s^2}{(s^2 + a^2)^3} ds = 2\pi i \left[ \operatorname{Res}_{s=s_1} e^{st} \frac{8a^3 s^2}{(s^2 + a^2)^3} + \operatorname{Res}_{s=s_2} e^{st} \frac{8a^3 s^2}{(s^2 + a^2)^3} \right]$$

The denominator can be factored as  $(s^2 + a^2)^3 = (s - s_1)^3 (s - s_2)^3$ . From this we see that the multiplicity of each factor is 3, so the residues are calculated by

$$\begin{aligned} \operatorname{Res}_{s=s_1} e^{st} \frac{8a^3 s^2}{(s^2 + a^2)^3} &= \frac{\phi_1^{(3-1)}(s_1)}{(3-1)!} = \frac{\phi_1''(s_1)}{2} \\ \operatorname{Res}_{s=s_2} e^{st} \frac{8a^3 s^2}{(s^2 + a^2)^3} &= \frac{\phi_2^{(3-1)}(s_2)}{(3-1)!} = \frac{\phi_2''(s_2)}{2}, \end{aligned}$$

where  $\phi_1(s)$  and  $\phi_2(s)$  are the same function as the integrand without the factors,  $s - s_1$  and  $s - s_2$ , respectively.

$$\phi_1(s) = e^{st} \frac{8a^3 s^2}{(s - s_2)^3}$$

$$\phi_2(s) = e^{st} \frac{8a^3 s^2}{(s - s_1)^3}$$

Use the quotient rule twice to obtain  $\phi_1''(s)$  and  $\phi_2''(s)$ .

$$\phi_1''(s) = \frac{8a^3 e^{st} [2(s^2 + 4s_2 s + s_2^2) - 2ts(s - s_2)(s + 2s_2) + s^2 t^2 (s - s_2)^2]}{(s - s_2)^5} \Rightarrow \frac{\phi_1''(s_1)}{2} = \frac{1}{2} [i(1 + a^2 t^2) - at] e^{-iat}$$

$$\phi_2''(s) = \frac{8a^3 e^{st} [2(s^2 + 4s_1 s + s_1^2) - 2ts(s - s_1)(s + 2s_1) + s^2 t^2 (s - s_1)^2]}{(s - s_1)^5} \Rightarrow \frac{\phi_2''(s_2)}{2} = \frac{1}{2} [-i(1 + a^2 t^2) - at] e^{iat}$$

So then

$$\begin{aligned} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} \frac{8a^3 s^2}{(s^2 + a^2)^3} ds &= 2\pi i \left\{ \frac{1}{2} [i(1 + a^2 t^2) - at] e^{-iat} + \frac{1}{2} [-i(1 + a^2 t^2) - at] e^{iat} \right\} \\ &= 2\pi i \left[ -i(1 + a^2 t^2) \frac{e^{iat} - e^{-iat}}{2} - at \frac{e^{iat} + e^{-iat}}{2} \right] \\ &= 2\pi i [-i(1 + a^2 t^2)(i \sin at) - at(\cos at)] \\ &= 2\pi i [(1 + a^2 t^2) \sin at - at \cos at]. \end{aligned}$$

Divide both sides by  $2\pi i$ .

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} \frac{8a^3 s^2}{(s^2 + a^2)^3} ds = (1 + a^2 t^2) \sin at - at \cos at$$

Therefore, the inverse Laplace transform of  $F(s)$  is

$$\boxed{f(t) = (1 + a^2 t^2) \sin at - at \cos at.}$$

The Integral Over  $C_R$ 

The objective here is to show that

$$\lim_{R \rightarrow \infty} \int_{C_R} e^{st} \frac{8a^3 s^2}{(s^2 + a^2)^3} ds = 0.$$

The parameterization of the semicircular arc with radius  $R$  in Figure 2 is  $s = Re^{i\theta}$ , where  $\theta$  goes from  $\pi/2$  to  $3\pi/2$ .

$$\begin{aligned} \int_{C_R} e^{st} \frac{8a^3 s^2}{(s^2 + a^2)^3} ds &= \int_{\pi/2}^{3\pi/2} e^{Re^{i\theta}t} \frac{8a^3 (Re^{i\theta})^2}{[(Re^{i\theta})^2 + a^2]^3} (Rie^{i\theta} d\theta) \\ &= \int_{\pi/2}^{3\pi/2} e^{R(\cos \theta + i \sin \theta)t} \frac{8ia^3 (Re^{i\theta})^3}{[(Re^{i\theta})^2 + a^2]^3} d\theta \\ &= \int_{\pi/2}^{3\pi/2} e^{Rt \cos \theta} e^{iRt \sin \theta} \frac{8ia^3 (Re^{i\theta})^3}{[(Re^{i\theta})^2 + a^2]^3} d\theta \end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned} \left| \int_{C_R} e^{st} \frac{8a^3 s^2}{(s^2 + a^2)^3} ds \right| &= \left| \int_{\pi/2}^{3\pi/2} e^{Rt \cos \theta} e^{iRt \sin \theta} \frac{8ia^3 (Re^{i\theta})^3}{[(Re^{i\theta})^2 + a^2]^3} d\theta \right| \\ &\leq \int_{\pi/2}^{3\pi/2} \left| e^{Rt \cos \theta} e^{iRt \sin \theta} \frac{8ia^3 (Re^{i\theta})^3}{[(Re^{i\theta})^2 + a^2]^3} \right| d\theta \\ &= \int_{\pi/2}^{3\pi/2} \left| e^{Rt \cos \theta} \right| \left| e^{iRt \sin \theta} \right| \frac{|8ia^3 (Re^{i\theta})^3|}{|[(Re^{i\theta})^2 + a^2]^3|} d\theta \\ &= \int_{\pi/2}^{3\pi/2} e^{Rt \cos \theta} \frac{8a^3 |Re^{i\theta}|^3}{|(Re^{i\theta})^2 + a^2|^3} d\theta \\ &= \int_{\pi/2}^{3\pi/2} e^{Rt \cos \theta} \frac{8a^3 R^3}{|(Re^{i\theta})^2 + a^2|^3} d\theta \\ &\leq \int_{\pi/2}^{3\pi/2} e^{Rt \cos \theta} \frac{8a^3 R^3}{(|Re^{i\theta}|^2 - |a^2|)^3} d\theta \\ &= \int_{\pi/2}^{3\pi/2} e^{Rt \cos \theta} \frac{8a^3 R^3}{(R^2 - a^2)^3} d\theta \\ &= \int_{\pi/2}^{3\pi/2} e^{Rt \cos \theta} \frac{\frac{8a^3}{R^3}}{(1 - \frac{a^2}{R^2})^3} d\theta \end{aligned}$$

So we have

$$\left| \int_{C_R} e^{st} \frac{8a^3 s^2}{(s^2 + a^2)^3} ds \right| \leq \int_{\pi/2}^{3\pi/2} e^{Rt \cos \theta} \frac{\frac{8a^3}{R^3}}{(1 - \frac{a^2}{R^2})^3} d\theta.$$

Take the limit of both sides now as  $R \rightarrow \infty$ .

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} e^{st} \frac{8a^3 s^2}{(s^2 + a^2)^3} ds \right| \leq \lim_{R \rightarrow \infty} \int_{\pi/2}^{3\pi/2} e^{Rt \cos \theta} \frac{\frac{8a^3}{R^3}}{(1 - \frac{a^2}{R^2})^3} d\theta$$

Because the limits of integration are constant, the limit may be brought inside the integral.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} e^{st} \frac{8a^3 s^2}{(s^2 + a^2)^3} ds \right| \leq \int_{\pi/2}^{3\pi/2} \lim_{R \rightarrow \infty} e^{Rt \cos \theta} \frac{\frac{8a^3}{R^3}}{\left(1 - \frac{a^2}{R^2}\right)^3} d\theta$$

Since  $\theta$  is between  $\pi/2$  and  $3\pi/2$ , the cosine of  $\theta$  is negative. In addition,  $R$  and  $t$  are positive, so the exponent of  $e$  tends to  $-\infty$ , which means the integral tends to zero.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} e^{st} \frac{8a^3 s^2}{(s^2 + a^2)^3} ds \right| \leq 0$$

The magnitude of a number cannot be negative.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} e^{st} \frac{8a^3 s^2}{(s^2 + a^2)^3} ds \right| = 0$$

The only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R \rightarrow \infty} \int_{C_R} e^{st} \frac{8a^3 s^2}{(s^2 + a^2)^3} ds = 0.$$