Exercise 11

Use residues to find the Cauchy principal values of the improper integrals in Exercises 9 through 11.

$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x+a)^2+b^2} \quad (b>0).$$

Solution

In order to evaluate the integral, consider the corresponding function in the complex plane,

$$f(z) = \frac{e^{iz}}{(z+a)^2+b^2},$$

and the contour in Fig. 93. Singularities occur where the denominator is equal to zero.

$$\begin{gathered}(z+a)^2+b^2=0\\ z+a=\pm ib\\ z=-a+ib \quad \text{or} \quad z=-a-ib\end{gathered}$$

The singular point of interest to us is the one that lies within the closed contour, $z=-a+ib$.

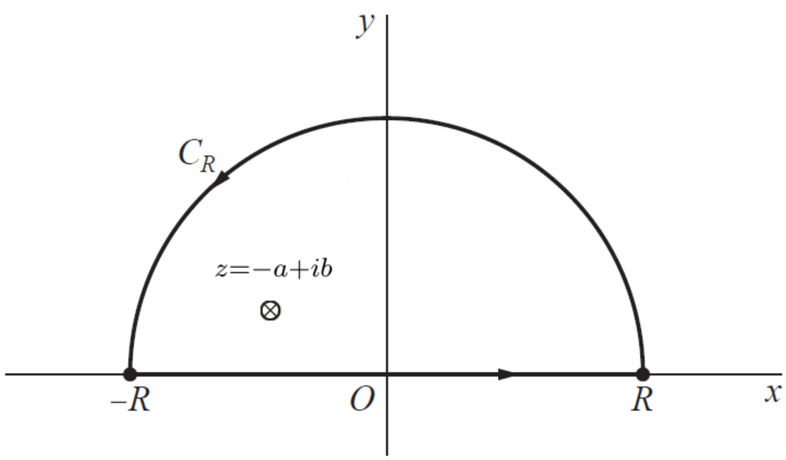

Figure 1: This is Fig. 93 with the singularity at $z=-a+ib$ marked.

According to Cauchy's residue theorem, the integral of $e^{iz}/[(z+a)^2+b^2]$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{e^{iz}}{(z+a)^2+b^2}\, dz = 2\pi i \operatorname*{Res}_{z=-a+ib} \frac{e^{iz}}{(z+a)^2+b^2}$$

This closed loop integral is the sum of two integrals, one over each arc in the loop.

$$\int_L \frac{e^{iz}}{(z+a)^2+b^2}\, dz + \int_{C_R} \frac{e^{iz}}{(z+a)^2+b^2}\, dz = 2\pi i \operatorname*{Res}_{z=-2+i} \frac{e^{iz}}{(z+a)^2+b^2}$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L: &\quad z=r, && r=-R \quad \to \quad r=R\\ C_R: &\quad z=Re^{i\theta}, && \theta=0 \quad \to \quad \theta=\pi \end{aligned}$$

As a result,

$$\int_{-R}^{R} \frac{e^{ir}}{(r+a)^2+b^2}\,dr + \int_{C_R} \frac{e^{iz}}{(z+a)^2+b^2}\,dz = 2\pi i \operatorname*{Res}_{z=-a+ib} \frac{e^{iz}}{(z+a)^2+b^2}.$$

Take the limit now as $R \to \infty$.

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ir}}{(r+a)^2+b^2}\,dr + \lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{(z+a)^2+b^2}\,dz = 2\pi i \operatorname*{Res}_{z=-a+ib} \frac{e^{iz}}{(z+a)^2+b^2}$$

If the first limit exists, then it's defined to be the Cauchy principal value (P.V.) of the integral. The second limit is zero. Proof for this statement will be given at the end.

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ir}}{(r+a)^2+b^2}\,dr = 2\pi i \operatorname*{Res}_{z=-a+ib} \frac{e^{iz}}{(z+a)^2+b^2}$$

The denominator can be written as $(z+a)^2+b^2 = [z-(-a+ib)][z-(-a-ib)]$. From this we see that the multiplicity of the $z-(-a+ib)$ factor is 1. The residue at $z=-a+ib$ can then be calculated by

$$\operatorname*{Res}_{z=-a+ib} \frac{e^{iz}}{(z+a)^2+b^2} = \phi(-a+ib),$$

where $\phi(z)$ is equal to $f(z)$ without the $z-(-a+ib)$ factor.

$$\phi(z) = \frac{e^{iz}}{z-(-a-ib)} \quad \Rightarrow \quad \phi(-a+ib) = \frac{1}{2ib}e^{-b-ia}$$

So then

$$\operatorname*{Res}_{z=-a+ib} \frac{e^{iz}}{(z+a)^2+b^2} = \frac{1}{2ib}e^{-b-ia}$$

and

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ir}}{(r+a)^2+b^2}\,dr = 2\pi i \left(\frac{1}{2ib}e^{-b-ia}\right)$$

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos r + i\sin r}{(r+a)^2+b^2}\,dr = \frac{\pi}{be^b}e^{-ia}$$

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos r}{(r+a)^2+b^2}\,dr + i\,\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin r}{(r+a)^2+b^2}\,dr = \frac{\pi}{be^b}\cos a - i\frac{\pi}{be^b}\sin a.$$

Match the real and imaginary parts of both sides.

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos r}{(r+a)^2+b^2}\,dr = \frac{\pi}{be^b}\cos a \quad \text{and} \quad \text{P.V.} \int_{-\infty}^{\infty} \frac{\sin r}{(r+a)^2+b^2}\,dr = -\frac{\pi}{be^b}\sin a$$

Therefore, changing the dummy integration variable to x,

$$\boxed{\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x+a)^2+b^2} = \frac{\pi}{be^b}\cos a, \quad b>0,\ a \text{ is real.}}$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \to \infty$. The parameterization of the semicircular arc in Fig. 93 is $z = Re^{i\theta}$, where θ goes from 0 to π.

$$\begin{aligned}
\int_{C_R} \frac{e^{iz}}{(z+a)^2+b^2}\,dz &= \int_0^\pi \frac{e^{iRe^{i\theta}}}{(Re^{i\theta}+a)^2+b^2}(Rie^{i\theta}\,d\theta) \\
&= \int_0^\pi \frac{e^{iR(\cos\theta+i\sin\theta)}}{[Re^{i\theta}-(-a+ib)][Re^{i\theta}-(-a-ib)]}(Rie^{i\theta}\,d\theta) \\
&= \int_0^\pi \frac{e^{iR\cos\theta}e^{-R\sin\theta}}{[Re^{i\theta}+(a-ib)][Re^{i\theta}+(a+ib)]}(Rie^{i\theta}\,d\theta)
\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}
\left|\int_{C_R} \frac{e^{iz}}{(z+a)^2+b^2}\,dz\right| &= \left|\int_0^\pi \frac{e^{iR\cos\theta}e^{-R\sin\theta}}{[Re^{i\theta}+(a-ib)][Re^{i\theta}+(a+ib)]}(Rie^{i\theta}\,d\theta)\right| \\
&\le \int_0^\pi \left|\frac{e^{iR\cos\theta}e^{-R\sin\theta}}{[Re^{i\theta}+(a-ib)][Re^{i\theta}+(a+ib)]}(Rie^{i\theta})\right|d\theta \\
&= \int_0^\pi \frac{\left|e^{iR\cos\theta}\right|\left|e^{-R\sin\theta}\right|}{|Re^{i\theta}+(a-ib)|\,|Re^{i\theta}+(a+ib)|}\left|Rie^{i\theta}\right|d\theta \\
&= \int_0^\pi \frac{e^{-R\sin\theta}}{|Re^{i\theta}+(a-ib)|\,|Re^{i\theta}+(a+ib)|}R\,d\theta \\
&\le \int_0^\pi \frac{e^{-R\sin\theta}}{(|Re^{i\theta}|-|a-ib|)(|Re^{i\theta}|-|a+ib|)}R\,d\theta \\
&= \int_0^\pi \frac{e^{-R\sin\theta}}{(R-\sqrt{a^2+b^2})^2}R\,d\theta \\
&= \int_0^\pi \frac{e^{-R\sin\theta}}{\left(1-\frac{\sqrt{a^2+b^2}}{R}\right)^2}\frac{d\theta}{R}
\end{aligned}$$

Now take the limit of both sides as $R \to \infty$.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{e^{iz}}{(z+a)^2+b^2}\,dz\right| \le \lim_{R\to\infty}\int_0^\pi \frac{e^{-R\sin\theta}}{\left(1-\frac{\sqrt{a^2+b^2}}{R}\right)^2}\frac{d\theta}{R}$$

Because the limits of integration do not depend on R, the limit may be brought inside the integral.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{e^{iz}}{(z+a)^2+b^2}\,dz\right| \le \int_0^\pi \lim_{R\to\infty}\frac{e^{-R\sin\theta}}{\left(1-\frac{\sqrt{a^2+b^2}}{R}\right)^2}\frac{d\theta}{R}$$

Since θ lies between 0 and π, the sine of θ is positive. Thus, the exponent of e tends to $-\infty$, and the integral tends to zero.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{e^{iz}}{(z+a)^2+b^2}\,dz\right| \le 0$$

The magnitude of a number cannot be negative, and the only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{e^{iz}}{(z+a)^2+b^2}\,dz\right| = 0 \quad \to \quad \lim_{R\to\infty}\int_{C_R} \frac{e^{iz}}{(z+a)^2+b^2}\,dz = 0.$$