Exercise 2

In Exercises 1 through 4, take the indented contour in Fig. 101 (Sec. 82).

Evaluate the improper integral

$$\int_0^\infty \frac{x^a}{(x^2+1)^2}\,dx, \quad \text{where} \quad -1 < a < 3 \text{ and } x^a = \exp(a \ln x).$$

$$Ans.\ \frac{(1-a)\pi}{4\cos(a\pi/2)}.$$

Solution

In order to evaluate the integral, consider the corresponding function in the complex plane, $z^a/(z^2+1)^2$, and the contour in Fig. 101. Singularities occur where the denominator is equal to zero.

$$\begin{aligned}(z^2+1)^2 &= 0\\ z^2+1 &= 0\\ z &= \pm i\end{aligned}$$

The singular point of interest to us is the one that lies within the closed contour, $z = i$. z^a can be written in terms of the logarithm function as

$$z^a = \exp(a \log z),$$

so a branch cut for the function has to be chosen. For convenience, we choose it to be the axis of negative imaginary numbers.

$$\begin{aligned} z^a &= \exp[a(\ln r + i\theta)], \quad \left(|z| > 0,\ -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right)\\ &= r^a e^{ia\theta},\end{aligned}$$

where $r = |z|$ is the magnitude of z. According to Cauchy's residue theorem, the integral of $z^a/(z^2+1)^2$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{z^a}{(z^2+1)^2}\,dz = 2\pi i \operatorname*{Res}_{z=i} \frac{z^a}{(z^2+1)^2}$$

This closed loop integral is the sum of four integrals, one over each arc in the loop.

$$\int_{L_1} \frac{z^a}{(z^2+1)^2}\,dz + \int_{C_\rho} \frac{z^a}{(z^2+1)^2}\,dz + \int_{L_2} \frac{z^a}{(z^2+1)^2}\,dz + \int_{C_R} \frac{z^a}{(z^2+1)^2}\,dz = 2\pi i \operatorname*{Res}_{z=i} \frac{z^a}{(z^2+1)^2}$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L_1 &: \quad z = re^{i0}, & r = \rho &\quad\to\quad r = R\\ L_2 &: \quad z = re^{i\pi}, & r = R &\quad\to\quad r = \rho\\ C_\rho &: \quad z = \rho e^{i\theta}, & \theta = \pi &\quad\to\quad \theta = 0\\ C_R &: \quad z = Re^{i\theta}, & \theta = 0 &\quad\to\quad \theta = \pi \end{aligned}$$

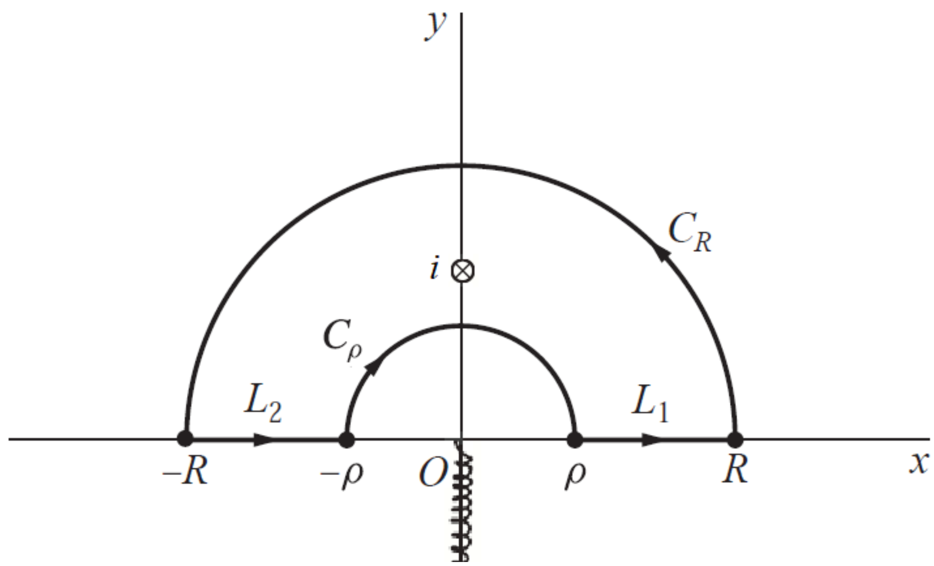

Figure 1: This is Fig. 101 with the singularity at $z = i$ marked. The squiggly line represents the branch cut ($|z| > 0,\ -\pi/2 < \theta < 3\pi/2$).

As a result,

$$\begin{aligned}
2\pi i \operatorname*{Res}_{z=i} \frac{z^a}{(z^2+1)^2} &= \int_\rho^R \frac{(re^{i0})^a}{[(re^{i0})^2+1]^2}(dr\, e^{i0}) + \int_{C_\rho} \frac{z^a}{(z^2+1)^2}\,dz + \int_R^\rho \frac{(re^{i\pi})^a}{[(re^{i\pi})^2+1]^2}(dr\, e^{i\pi}) + \int_{C_R} \frac{z^a}{(z^2+1)^2}\,dz \\
&= \int_\rho^R \frac{r^a}{(r^2+1)^2}\,dr + \int_{C_\rho} \frac{z^a}{(z^2+1)^2}\,dz + \int_R^\rho \frac{r^a e^{ia\pi}}{[(-r)^2+1]^2}(-dr) + \int_{C_R} \frac{z^a}{(z^2+1)^2}\,dz \\
&= \int_\rho^R \frac{r^a}{(r^2+1)^2}\,dr + \int_{C_\rho} \frac{z^a}{(z^2+1)^2}\,dz + \int_\rho^R \frac{r^a e^{ia\pi}}{(r^2+1)^2}\,dr + \int_{C_R} \frac{z^a}{(z^2+1)^2}\,dz \\
&= \int_\rho^R \frac{r^a + r^a e^{ia\pi}}{(r^2+1)^2}\,dr + \int_{C_\rho} \frac{z^a}{(z^2+1)^2}\,dz + \int_{C_R} \frac{z^a}{(z^2+1)^2}\,dz \\
&= (1+e^{ia\pi})\int_\rho^R \frac{r^a}{(r^2+1)^2}\,dr + \int_{C_\rho} \frac{z^a}{(z^2+1)^2}\,dz + \int_{C_R} \frac{z^a}{(z^2+1)^2}\,dz.
\end{aligned}$$

Take the limit now as $\rho \to 0$ and $R \to \infty$. The integral over C_ρ tends to zero, and the integral over C_R tends to zero. Proof for these statements will be given at the end.

$$(1+e^{ia\pi})\int_0^\infty \frac{r^a}{(r^2+1)^2}\,dr = 2\pi i \operatorname*{Res}_{z=i} \frac{z^a}{(z^2+1)^2}$$

The denominator can be written as $(z^2+1)^2 = (z+i)^2(z-i)^2$. From this we see that the multiplicity of the factor $z - i$ is 2. The residue at $z = i$ can then be calculated by

$$\operatorname*{Res}_{z=i} \frac{z^a}{(z^2+1)^2} = \frac{\phi^{(2-1)}(i)}{(2-1)!} = \phi'(i),$$

where $\phi(z)$ is the same function as the integrand without $(z-i)^2$.

$$\phi(z) = \frac{z^a}{(z+i)^2}$$

Calculate its derivative using the quotient rule.

$$\begin{aligned}\phi'(z) &= \frac{az^{a-1}(z+i)^2 - 2(z+i)z^a}{(z+i)^4} \\ &= \frac{az^{a-1}(z+i) - 2z^a}{(z+i)^3}\end{aligned}$$

So then

$$\phi'(i) = \frac{ai^{a-1}(2i) - 2i^a}{(2i)^3} = \frac{2ai^a - 2i^a}{8i^3} = \frac{2i^a(a-1)}{-8i} = \frac{i^a(1-a)}{4i}$$

and

$$\operatorname*{Res}_{z=i} \frac{z^a}{(z^2+1)^2} = \frac{i^a(1-a)}{4i}.$$

Consequently,

$$\begin{aligned}(1+e^{ia\pi}) \int_0^\infty \frac{r^a}{(r^2+1)^2}\,dr &= 2\pi i \left[\frac{i^a(1-a)}{4i}\right] \\ &= \frac{i^a(1-a)\pi}{2}.\end{aligned}$$

Use the fact that $i = e^{i\pi/2}$ to write $i^a = e^{ia\pi/2}$.

$$(1+e^{ia\pi}) \int_0^\infty \frac{r^a}{(r^2+1)^2}\,dr = \frac{e^{ia\pi/2}(1-a)\pi}{2}$$

Divide both sides by $1 + e^{ia\pi}$.

$$\begin{aligned}\int_0^\infty \frac{r^a}{(r^2+1)^2}\,dr &= \frac{(1-a)\pi}{2} \frac{e^{ia\pi/2}}{1+e^{ia\pi}} \\ &= \frac{(1-a)\pi}{2} \frac{1}{e^{-ia\pi/2}+e^{ia\pi/2}} \\ &= \frac{(1-a)\pi}{2} \frac{1}{2\cos(a\pi/2)}\end{aligned}$$

Change the dummy variable of integration to x. Therefore,

$$\boxed{\int_0^\infty \frac{x^a}{(x^2+1)^2}\,dx = \frac{(1-a)\pi}{4\cos(a\pi/2)}.}$$

The Integral Over C_ρ

Our aim here is to show that the integral over C_ρ tends to zero in the limit as $\rho \to 0$. The parameterization of the small semicircular arc in Fig. 101 is $z = \rho e^{i\theta}$, where θ goes from π to 0.

$$\begin{aligned}\int_{C_\rho} \frac{z^a}{(z^2+1)^2}\,dz &= \int_\pi^0 \frac{(\rho e^{i\theta})^a}{[(\rho e^{i\theta})^2+1]^2}(\rho i e^{i\theta}\,d\theta) \\ &= \int_\pi^0 \frac{\rho^{1+a}e^{i\theta(1+a)}}{(\rho^2 e^{2i\theta}+1)^2}(i\,d\theta)\end{aligned}$$

In the limit as $\rho \to 0$, we have

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{z^a}{(z^2+1)^2}\,dz = \lim_{\rho\to 0}\int_\pi^0 \frac{\rho^{1+a}e^{i\theta(1+a)}}{(\rho^2 e^{2i\theta}+1)^2}(i\,d\theta).$$

Because the limits of integration are constant, the limit may be brought inside the integral.

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{z^a}{(z^2+1)^2}\,dz = \int_\pi^0 \lim_{\rho\to 0}\frac{\rho^{1+a}e^{i\theta(1+a)}}{(\rho^2 e^{2i\theta}+1)^2}(i\,d\theta)$$

Since $-1 < a < 3$, $1 + a$ is positive, and ρ^{1+a} tends to zero as a result. Therefore,

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{z^a}{(z^2+1)^2}\,dz = 0.$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \to \infty$. The parameterization of the large semicircular arc in Fig. 101 is $z = Re^{i\theta}$, where θ goes from 0 to π.

$$\begin{aligned}\int_{C_R} \frac{z^a}{(z^2+1)^2}\,dz &= \int_0^\pi \frac{(Re^{i\theta})^a}{[(Re^{i\theta})^2+1]^2}(Rie^{i\theta}\,d\theta) \\ &= \int_0^\pi \frac{R^{1+a}e^{i\theta(1+a)}}{(R^2 e^{2i\theta}+1)^2}(i\,d\theta)\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}\left|\int_{C_R} \frac{z^a}{(z^2+1)^2}\,dz\right| &= \left|\int_0^\pi \frac{R^{1+a}e^{i\theta(1+a)}}{(R^2 e^{2i\theta}+1)^2}(i\,d\theta)\right| \\ &\le \int_0^\pi \left|\frac{R^{1+a}e^{i\theta(1+a)}}{(R^2 e^{2i\theta}+1)^2}(i)\right| d\theta \\ &= \int_0^\pi \frac{\left|R^{1+a}e^{i\theta(1+a)}\right|}{\left|(R^2 e^{2i\theta}+1)^2\right|}\,|i|\,d\theta \\ &= \int_0^\pi \frac{R^{1+a}}{\left|R^2 e^{2i\theta}+1\right|^2}\,d\theta \\ &\le \int_0^\pi \frac{R^{1+a}}{(|R^2 e^{2i\theta}|-|1|)^2}\,d\theta \\ &= \int_0^\pi \frac{R^{1+a}}{(R^2-1)^2}\,d\theta\end{aligned}$$

So we have

$$\begin{aligned}\left|\int_{C_R} \frac{z^a}{(z^2+1)^2}\,dz\right| &\leq \frac{R^{1+a}}{(R^2-1)^2}\pi \\ &= \frac{R^{1+a}}{R^4\left(1-\frac{1}{R^2}\right)^2}\pi \\ &= \frac{\pi}{R^{3-a}\left(1-\frac{1}{R^2}\right)^2}.\end{aligned}$$

Take the limit of both sides as $R \to \infty$.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{z^a}{(z^2+1)^2}\,dz\right| \leq \lim_{R\to\infty} \frac{\pi}{R^{3-a}\left(1-\frac{1}{R^2}\right)^2}$$

Since $-1 < a < 3$, $3-a$ is positive, and the denominator tends to infinity as a result.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{z^a}{(z^2+1)^2}\,dz\right| \leq 0$$

The magnitude of a number cannot be negative.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{z^a}{(z^2+1)^2}\,dz\right| = 0$$

The only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R\to\infty}\int_{C_R} \frac{z^a}{(z^2+1)^2}\,dz = 0.$$