Exercise 4

In Exercises 1 through 4, take the indented contour in Fig. 101 (Sec. 82).

Use the function

$$f(z) = \frac{(\log z)^2}{z^2+1} \qquad \left(|z| > 0,\ -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

to show that

$$\int_0^\infty \frac{(\ln x)^2}{x^2+1}\,dx = \frac{\pi^3}{8}, \qquad \int_0^\infty \frac{\ln x}{x^2+1}\,dx = 0.$$

Suggestion: The integration formula obtained in Exercise 1, Sec. 79, is needed here.

Solution

In order to evaluate these integrals, consider the given function in the complex plane and the contour in Fig. 101. Singularities occur where the denominator is equal to zero.

$$z^2 + 1 = 0$$
$$z = \pm i$$

The singular point of interest to us is the one that lies within the closed contour, $z = i$. The branch cut for the logarithm function has been conveniently chosen to be the axis of negative imaginary numbers.

$$\log z = \ln r + i\theta, \quad \left(|z| > 0,\ -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right),$$

where $r = |z|$ is the magnitude of z and $\theta = \arg z$ is the argument of z.

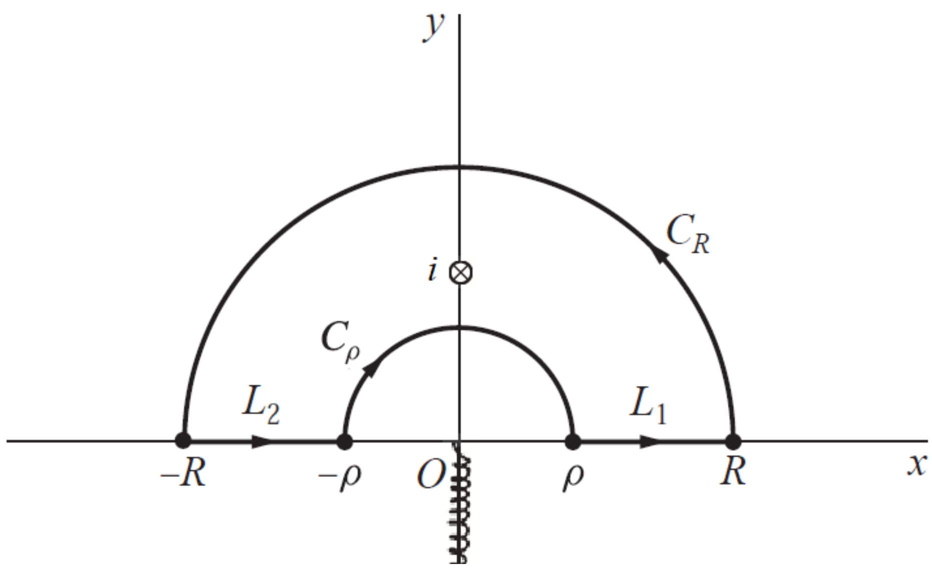

Figure 1: This is Fig. 101 with the singularity at $z = i$ marked. The squiggly line represents the branch cut ($|z| > 0,\ -\pi/2 < \theta < 3\pi/2$).

According to Cauchy's residue theorem, the integral of $(\log z)^2/(z^2+1)$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{(\log z)^2}{z^2+1}\,dz = 2\pi i \operatorname*{Res}_{z=i} \frac{(\log z)^2}{z^2+1}$$

This closed loop integral is the sum of four integrals, one over each arc in the loop.

$$\int_{L_1} \frac{(\log z)^2}{z^2+1}\,dz + \int_{L_2} \frac{(\log z)^2}{z^2+1}\,dz + \int_{C_\rho} \frac{(\log z)^2}{z^2+1}\,dz + \int_{C_R} \frac{(\log z)^2}{z^2+1}\,dz = 2\pi i \operatorname*{Res}_{z=i} \frac{(\log z)^2}{z^2+1}$$

The parameterizations for the arcs are as follows.

$$\begin{array}{lll} L_1: & z = re^{i0}, & r=\rho \quad \to \quad r=R \\ L_2: & z = re^{i\pi}, & r=R \quad \to \quad r=\rho \\ C_\rho: & z=\rho e^{i\theta}, & \theta=\pi \quad \to \quad \theta=0 \\ C_R: & z=Re^{i\theta}, & \theta=0 \quad \to \quad \theta=\pi \end{array}$$

As a result,

$$\begin{aligned} 2\pi i \operatorname*{Res}_{z=i} \frac{(\log z)^2}{z^2+1} &= \int_\rho^R \frac{[\log(re^{i0})]^2}{(re^{i0})^2+1}(dr\, e^{i0}) + \int_R^\rho \frac{[\log(re^{i\pi})]^2}{(re^{i\pi})^2+1}(dr\, e^{i\pi}) + \int_{C_\rho} \frac{(\log z)^2}{z^2+1}\,dz + \int_{C_R} \frac{(\log z)^2}{z^2+1}\,dz \\ &= \int_\rho^R \frac{(\ln r + i0)^2}{r^2+1}\,dr + \int_R^\rho \frac{(\ln r + i\pi)^2}{(-r)^2+1}(-dr) + \int_{C_\rho} \frac{(\log z)^2}{z^2+1}\,dz + \int_{C_R} \frac{(\log z)^2}{z^2+1}\,dz \\ &= \int_\rho^R \frac{(\ln r)^2}{r^2+1}\,dr + \int_\rho^R \frac{(\ln r)^2 + 2i\pi \ln r - \pi^2}{r^2+1}\,dr + \int_{C_\rho} \frac{(\log z)^2}{z^2+1}\,dz + \int_{C_R} \frac{(\log z)^2}{z^2+1}\,dz \\ &= 2\int_\rho^R \frac{(\ln r)^2}{r^2+1}\,dr - \pi^2 \int_\rho^R \frac{dr}{r^2+1} + 2i\pi \int_\rho^R \frac{\ln r}{r^2+1}\,dr + \int_{C_\rho} \frac{(\log z)^2}{z^2+1}\,dz + \int_{C_R} \frac{(\log z)^2}{z^2+1}\,dz. \end{aligned}$$

Take the limit now as $\rho \to 0$ and $R \to \infty$. The integral over C_ρ tends to zero, and the integral over C_R tends to zero. Proof for these statements will be given at the end.

$$2\int_0^\infty \frac{(\ln r)^2}{r^2+1}\,dr - \pi^2 \int_0^\infty \frac{dr}{r^2+1} + 2i\pi \int_0^\infty \frac{\ln r}{r^2+1}\,dr = 2\pi i \operatorname*{Res}_{z=i} \frac{(\log z)^2}{z^2+1}$$

Evaluate the integral without $\ln r$.

$$2\int_0^\infty \frac{(\ln r)^2}{r^2+1}\,dr - \pi^2 \tan^{-1} r\Big|_0^\infty + 2i\pi \int_0^\infty \frac{\ln r}{r^2+1}\,dr = 2\pi i \operatorname*{Res}_{z=i} \frac{(\log z)^2}{z^2+1}$$

$$2\int_0^\infty \frac{(\ln r)^2}{r^2+1}\,dr - \pi^2 \left(\frac{\pi}{2}\right) + 2i\pi \int_0^\infty \frac{\ln r}{r^2+1}\,dr = 2\pi i \operatorname*{Res}_{z=i} \frac{(\log z)^2}{z^2+1}$$

The denominator can be written as $z^2+1 = (z+i)(z-i)$. From this we see that the multiplicity of the factor $z-i$ is 1. The residue at $z=i$ can then be calculated by

$$\operatorname*{Res}_{z=i} \frac{(\log z)^2}{z^2+1} = \phi(i),$$

where $\phi(z)$ is the same function as $f(z)$ without $(z-i)$.

$$\phi(z) = \frac{(\log z)^2}{z+i} \quad \Rightarrow \quad \phi(i) = \frac{(\log i)^2}{2i} = \frac{\left(\ln 1 + i\frac{\pi}{2}\right)^2}{2i} = -\frac{\pi^2}{8i}$$

So then

$$\operatorname*{Res}_{z=i} \frac{(\log z)^2}{z^2+1} = -\frac{\pi^2}{8i}.$$

and

$$2\int_0^\infty \frac{(\ln r)^2}{r^2+1}\,dr - \pi^2\left(\frac{\pi}{2}\right) + 2i\pi\int_0^\infty \frac{\ln r}{r^2+1}\,dr = 2\pi i\left(-\frac{\pi^2}{8i}\right)$$

$$2\int_0^\infty \frac{(\ln r)^2}{r^2+1}\,dr - \frac{\pi^3}{2} + 2i\pi\int_0^\infty \frac{\ln r}{r^2+1}\,dr = -\frac{\pi^3}{4}$$

$$2\int_0^\infty \frac{(\ln r)^2}{r^2+1}\,dr + 2i\pi\int_0^\infty \frac{\ln r}{r^2+1}\,dr = \frac{\pi^3}{4}$$

$$\int_0^\infty \frac{(\ln r)^2}{r^2+1}\,dr + i\pi\int_0^\infty \frac{\ln r}{r^2+1}\,dr = \frac{\pi^3}{8}.$$

Match the real and imaginary parts of both sides of the equation.

$$\int_0^\infty \frac{(\ln r)^2}{r^2+1}\,dr = \frac{\pi^3}{8}$$

$$\pi\int_0^\infty \frac{\ln r}{r^2+1}\,dr = 0$$

Therefore, changing the dummy integration variables to x,

$$\boxed{\int_0^\infty \frac{(\ln x)^2}{x^2+1}\,dx = \frac{\pi^3}{8}}$$

and

$$\boxed{\int_0^\infty \frac{\ln x}{x^2+1}\,dx = 0.}$$

The Integral Over C_ρ

Our aim here is to show that the integral over C_ρ tends to zero in the limit as $\rho \to 0$. The parameterization of the small semicircular arc in Fig. 101 is $z = \rho e^{i\theta}$, where θ goes from π to 0.

$$\begin{aligned}
\int_{C_\rho} \frac{(\log z)^2}{z^2+1}\,dz &= \int_\pi^0 \frac{[\log(\rho e^{i\theta})]^2}{(\rho e^{i\theta})^2+1}(\rho i e^{i\theta}\,d\theta) \\
&= \int_\pi^0 \frac{(\ln\rho + i\theta)^2}{\rho^2 e^{2i\theta}+1}(\rho i e^{i\theta}\,d\theta) \\
&= \int_\pi^0 \frac{\left(1+\frac{i\theta}{\ln\rho}\right)^2}{\rho^2 e^{2i\theta}+1}\rho(\ln\rho)^2(ie^{i\theta}\,d\theta)
\end{aligned}$$

In the limit as $\rho \to 0$, we have

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{(\log z)^2}{z^2+1}\,dz = \lim_{\rho\to 0}\int_\pi^0 \frac{\left(1+\frac{i\theta}{\ln\rho}\right)^2}{\rho^2 e^{2i\theta}+1}\rho(\ln\rho)^2(ie^{i\theta}\,d\theta).$$

Because the limits of integration are constant, the limit may be brought inside the integral.

$$\begin{aligned}
\lim_{\rho\to 0}\int_{C_\rho} \frac{(\log z)^2}{z^2+1}\,dz &= \int_\pi^0 \lim_{\rho\to 0}\frac{\left(1+\frac{i\theta}{\ln\rho}\right)^2}{\rho^2 e^{2i\theta}+1}\rho(\ln\rho)^2(ie^{i\theta}\,d\theta) \\
&= \int_\pi^0 \left[\lim_{\rho\to 0}\frac{\left(1+\frac{i\theta}{\ln\rho}\right)^2}{\rho^2 e^{2i\theta}+1}\right]\left[\lim_{\rho\to 0}\rho(\ln\rho)^2\right](ie^{i\theta}\,d\theta) \\
&= \int_\pi^0 [1]\left[\lim_{\rho\to 0}\frac{(\ln\rho)^2}{\rho^{-1}}\right](ie^{i\theta}\,d\theta)
\end{aligned}$$

Plugging in $\rho = 0$ results in the indeterminate form ∞/∞, so l'Hôpital's rule will be applied to calculate the limit.

$$\begin{aligned}
&\overset{\frac{\infty}{\infty}}{\underset{\text{H}}{=}} \int_\pi^0 \left[\lim_{\rho\to 0}\frac{2(\ln\rho)\frac{1}{\rho}}{-\rho^{-2}}\right](ie^{i\theta}\,d\theta) \\
&= \int_\pi^0 \left[2\lim_{\rho\to 0}\frac{\ln\rho}{-\rho^{-1}}\right](ie^{i\theta}\,d\theta)
\end{aligned}$$

Apply l'Hôpital's rule once more.

$$\begin{aligned}
&\overset{\frac{\infty}{\infty}}{\underset{\text{H}}{=}} \int_\pi^0 \left[2\lim_{\rho\to 0}\frac{\frac{1}{\rho}}{\rho^{-2}}\right](ie^{i\theta}\,d\theta) \\
&= \int_\pi^0 \left[2\lim_{\rho\to 0}\rho\right](ie^{i\theta}\,d\theta) \\
&= 0
\end{aligned}$$

Therefore,

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{(\log z)^2}{z^2+1}\,dz = 0.$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \to \infty$. The parameterization of the large semicircular arc in Fig. 101 is $z = Re^{i\theta}$, where θ goes from 0 to π.

$$\begin{aligned}\int_{C_R} \frac{(\log z)^2}{z^2+1}\,dz &= \int_0^\pi \frac{[\log(Re^{i\theta})]^2}{(Re^{i\theta})^2+1}(Rie^{i\theta}\,d\theta) \\ &= \int_0^\pi \frac{(\ln R + i\theta)^2}{R^2e^{2i\theta}+1}(Rie^{i\theta}\,d\theta)\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}\left|\int_{C_R} \frac{(\log z)^2}{z^2+1}\,dz\right| &= \left|\int_0^\pi \frac{(\ln R + i\theta)^2}{R^2e^{2i\theta}+1}(Rie^{i\theta}\,d\theta)\right| \\ &\leq \int_0^\pi \left|\frac{(\ln R + i\theta)^2}{R^2e^{2i\theta}+1}(Rie^{i\theta})\right| d\theta \\ &= \int_0^\pi \frac{\left|(\ln R + i\theta)^2\right|}{|R^2e^{2i\theta}+1|}\left|Rie^{i\theta}\right| d\theta \\ &= \int_0^\pi \frac{|\ln R + i\theta|^2}{|R^2e^{2i\theta}+1|}R\,d\theta \\ &\leq \int_0^\pi \frac{(|\ln R| + |i\theta|)^2}{|R^2e^{2i\theta}| - |1|}R\,d\theta \\ &= \int_0^\pi \frac{(\ln R + \theta)^2}{R^2-1}R\,d\theta \\ &= \int_0^\pi \frac{(\ln R)^2(1+\frac{\theta}{\ln R})^2}{R^2\left(1-\frac{1}{R^2}\right)}R\,d\theta\end{aligned}$$

So we have

$$\left|\int_{C_R} \frac{(\log z)^2}{z^2+1}\,dz\right| \leq \int_0^\pi \frac{(1+\frac{\theta}{\ln R})^2}{\left(1-\frac{1}{R^2}\right)}\frac{(\ln R)^2}{R}\,d\theta.$$

Take the limit of both sides as $R \to \infty$.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{(\log z)^2}{z^2+1}\,dz\right| \leq \lim_{R\to\infty}\int_0^\pi \frac{(1+\frac{\theta}{\ln R})^2}{\left(1-\frac{1}{R^2}\right)}\frac{(\ln R)^2}{R}\,d\theta$$

Because the limits of integration are constant, the limit may be brought inside the integral.

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{C_R} \frac{(\log z)^2}{z^2+1}\,dz\right| &\leq \int_0^\pi \lim_{R\to\infty}\frac{(1+\frac{\theta}{\ln R})^2}{\left(1-\frac{1}{R^2}\right)}\frac{(\ln R)^2}{R}\,d\theta \\ &= \int_0^\pi \left[\lim_{R\to\infty}\frac{(1+\frac{\theta}{\ln R})^2}{\left(1-\frac{1}{R^2}\right)}\right]\left[\lim_{R\to\infty}\frac{(\ln R)^2}{R}\right]d\theta\end{aligned}$$

The second limit is the indeterminate form ∞/∞, so l'Hôpital's rule will be applied to calculate it.

$$\overset{\frac{\infty}{\infty}}{\underset{\text{H}}{=}} \int_0^\pi [1]\left[\lim_{R\to\infty}\frac{2(\ln R)\frac{1}{R}}{1}\right]d\theta$$

$$\lim_{R\to\infty}\left|\int_{C_R}\frac{(\log z)^2}{z^2+1}\,dz\right| \leq \int_0^\pi \left[2\lim_{R\to\infty}\frac{\ln R}{R}\right]d\theta$$

Apply l'Hôpital's rule once more.

$$\begin{aligned}&\overset{\infty/\infty}{\underset{\text{H}}{=}} \int_0^\pi \left[2\lim_{R\to\infty}\frac{\frac{1}{R}}{1}\right]d\theta \\ &= \int_0^\pi \left[2\lim_{R\to\infty}\frac{1}{R}\right]d\theta \\ &= 0\end{aligned}$$

So we have

$$\lim_{R\to\infty}\left|\int_{C_R}\frac{(\log z)^2}{z^2+1}\,dz\right| \leq 0.$$

The magnitude of a number cannot be negative.

$$\lim_{R\to\infty}\left|\int_{C_R}\frac{(\log z)^2}{z^2+1}\,dz\right| = 0$$

The only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R\to\infty}\int_{C_R}\frac{(\log z)^2}{z^2+1}\,dz = 0.$$