Exercise 2

Find the three cube roots c_k $(k = 0, 1, 2)$ of $-8i$, express them in rectangular coordinates, and point out why they are as shown in Fig. 15.

$$Ans.\ \pm\sqrt{3} - i, 2i.$$

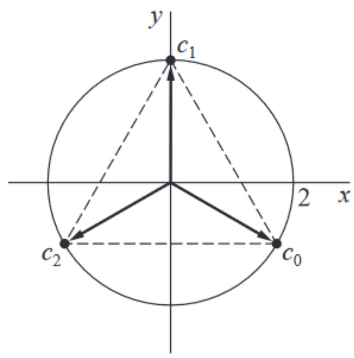

FIGURE 15

Solution

For a nonzero complex number $z = re^{i(\Theta+2\pi k)}$, its nth roots are

$$z^{1/n} = \left[re^{i(\Theta+2\pi k)}\right]^{1/n} = r^{1/n}\exp\left(i\frac{\Theta + 2\pi k}{n}\right), \quad k = 0, 1, 2, \ldots, n-1.$$

The magnitude of $-8i$ is 8, and the principal argument is $-\pi/2$.

$$(-8i)^{1/3} = 8^{1/3}\exp\left(i\frac{-\pi/2 + 2\pi k}{3}\right), \quad k = 0, 1, 2$$

The first, or principal, root $(k = 0)$ is

$$(-8i)^{1/3} = 8^{1/3}e^{-i\pi/6} = 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = \sqrt{3} - i = c_0,$$

the second root $(k = 1)$ is

$$(-8i)^{1/3} = 8^{1/3}e^{i\pi/2} = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2(0 + i) = 2i = c_1,$$

and the third root $(k = 2)$ is

$$(-8i)^{1/3} = 8^{1/3}e^{i7\pi/6} = 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = -\sqrt{3} - i = c_2.$$

Plotting each of these roots on the complex plane gives Fig. 15.