Exercise 3

Find $(-8-8\sqrt{3}i)^{1/4}$, express the roots in rectangular coordinates, exhibit them as the vertices of a certain square, and point out which is the principal root.

$$Ans.\ \pm(\sqrt{3}-i),\ \pm(1+\sqrt{3}i).$$

Solution

For a nonzero complex number $z = re^{i(\Theta+2\pi k)}$, its fourth roots are

$$z^{1/4} = \left[re^{i(\Theta+2\pi k)}\right]^{1/4} = r^{1/4}\exp\left(i\frac{\Theta+2\pi k}{4}\right), \quad k = 0, 1, 2, 3.$$

The magnitude and principal argument of $-8-8\sqrt{3}i$ are respectively

$$r = \sqrt{(-8)^2+(-8\sqrt{3})^2} = 16 \quad \text{and} \quad \Theta = \tan^{-1}\frac{-8\sqrt{3}}{-8} - \pi = -\frac{2\pi}{3},$$

so

$$(-8-8\sqrt{3}i)^{1/4} = 16^{1/4}\exp\left(i\frac{-\frac{2\pi}{3}+2\pi k}{4}\right), \quad k = 0, 1, 2, 3.$$

The first, or principal, root $(k=0)$ is

$$(-8-8\sqrt{3}i)^{1/4} = 16^{1/4}e^{-i\pi/6} = 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = \sqrt{3}-i,$$

the second root $(k=1)$ is

$$(-8-8\sqrt{3}i)^{1/4} = 16^{1/4}e^{i\pi/3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 1+\sqrt{3}i,$$

the third root $(k=2)$ is

$$(-8-8\sqrt{3}i)^{1/4} = 16^{1/4}e^{i5\pi/6} = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = -\sqrt{3}+i,$$

and the fourth root $(k=3)$ is

$$(-8-8\sqrt{3}i)^{1/4} = 16^{1/4}e^{i4\pi/3} = 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = 2\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -1-\sqrt{3}i.$$

The roots are drawn below in the complex plane.