Exercise 7

Show that if c is any nth root of unity other than unity itself, then

$$1 + c + c^2 + \cdots + c^{n-1} = 0.$$

Suggestion: Use the first identity in Exercise 9, Sec. 9.

Solution

The first identity in Exercise 9 of Sec. 9 is

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \qquad (z \neq 1).$$

Suppose that c is any nth root of unity other than unity itself.

$$c^n = 1, \quad (c \neq 1)$$

This root satisfies the previous identity.

$$1 + c + c^2 + \cdots + c^{n-1} + c^n = \frac{1 - c^{n+1}}{1 - c}$$

Subtract c^n from both sides.

$$\begin{aligned} 1 + c + c^2 + \cdots + c^{n-1} &= \frac{1 - c^{n+1}}{1 - c} - c^n \\ &= \frac{1 - c^{n+1} - c^n(1 - c)}{1 - c} \\ &= \frac{1 - c^{n+1} - c^n + c^{n+1}}{1 - c} \\ &= \frac{1 - c^n}{1 - c} \\ &= \frac{1 - 1}{1 - c} \\ &= 0 \end{aligned}$$