Exercise 8

(a) Prove that the usual formula solves the quadratic equation

$$az^2 + bz + c = 0 \qquad (a \neq 0)$$

when the coefficients a, b, and c are complex numbers. Specifically, by completing the square on the left-hand side, derive the ***quadratic formula***

$$z = \frac{-b + (b^2 - 4ac)^{1/2}}{2a},$$

where both square roots are to be considered when $b^2 - 4ac \neq 0$,

(b) Use the result in part (a) to find the roots of the equation $z^2 + 2z + (1 - i) = 0$.

$$\textit{Ans.}\ (b)\ \left(-1 + \frac{1}{\sqrt{2}}\right) + \frac{i}{\sqrt{2}}, \quad \left(-1 - \frac{1}{\sqrt{2}}\right) - \frac{i}{\sqrt{2}}.$$

[A period is needed here instead of a comma.]

Solution

Part (a)

$$az^2 + bz + c = 0 \qquad (a \neq 0)$$

To complete the square, begin by dividing both sides by a.

$$z^2 + \frac{b}{a}z + \frac{c}{a} = 0$$

Add $(b^2/4a^2)$ to both sides, completing the square.

$$z^2 + \frac{b}{a}z + \frac{b^2}{4a^2} + \frac{c}{a} = \frac{b^2}{4a^2}$$

The first three terms on the left can be factored like so.

$$\left(z + \frac{b}{2a}\right)^2 + \frac{c}{a} = \frac{b^2}{4a^2}$$

Subtract both sides by c/a.

$$\begin{aligned}\left(z + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{c}{a} \\ &= \frac{b^2 - 4ac}{4a^2}\end{aligned}$$

Multiply both sides by $4a^2$.

$$4a^2\left(z + \frac{b}{2a}\right)^2 = b^2 - 4ac$$

Bring $4a^2$ inside the parentheses.

$$\left[2a\left(z+\frac{b}{2a}\right)\right]^2 = b^2-4ac$$

$$(2az+b)^2 = b^2-4ac$$

Consider the square root of $2az+b$ and then solve for z.

$$2az+b = (b^2-4ac)^{1/2}$$

$$2az = -b+(b^2-4ac)^{1/2}$$

Therefore,

$$z = \frac{-b+(b^2-4ac)^{1/2}}{2a}.$$

Part (b)

We will solve $z^2+2z+(1-i)=0$ for z using the result from part (a). Here $a=1$, $b=2$, and $c=1-i$.

$$\begin{aligned} z &= \frac{-2+[2^2-4(1-i)]^{1/2}}{2} \\ &= \frac{-2+(4-4+4i)^{1/2}}{2} \\ &= \frac{-2+(4i)^{1/2}}{2} \end{aligned}$$

The magnitude and principal argument of $4i$ are respectively $r=4$ and $\Theta=\pi/2$. As a result, the square roots of $4i$ are

$$(4i)^{1/2} = 4^{1/2}\exp\left(i\frac{\frac{\pi}{2}+2\pi k}{2}\right), \quad k=0,1.$$

The first one $(k=0)$ is

$$(4i)^{1/2} = 4^{1/2}e^{i\pi/4} = 2\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right) = \sqrt{2}(1+i),$$

and the second one $(k=1)$ is

$$(4i)^{1/2} = 4^{1/2}e^{i5\pi/4} = 2\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right) = 2\left(-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right) = -\sqrt{2}(1+i).$$

Therefore,

$$z = \frac{-2+\sqrt{2}(1+i)}{2} \quad \text{or} \quad z = \frac{-2-\sqrt{2}(1+i)}{2}$$

$$z = -1+\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}} \quad \text{or} \quad z = -1-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}.$$