Exercise 11

Solve the equation $z^2 + z + 1 = 0$ for $z = (x, y)$ by writing

$$(x, y)(x, y) + (x, y) + (1, 0) = (0, 0)$$

and then solving a pair of simultaneous equations in x and y.

Suggestion: Use the fact that no real number x satisfies the given equation to show that $y \neq 0$.

$$\textit{Ans. } z = \left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right).$$

Solution

Use the definition of multiplication of complex numbers in equation (4) on page 2.

$$\begin{aligned} 0 &= z^2 + z + 1 \\ (0, 0) &= (x, y)(x, y) + (x, y) + (1, 0) \\ &= (x^2 - y^2, yx + xy) + (x, y) + (1, 0) \\ &= (x^2 - y^2, 2xy) + (x, y) + (1, 0) \\ &= (x^2 + x + 1 - y^2, 2xy + y) \end{aligned}$$

Match the real and imaginary parts on both sides to get a system of equations for x and y.

$$\left.\begin{aligned} x^2 + x + 1 - y^2 = 0 \\ 2xy + y = 0 \end{aligned}\right\}$$

Factor the second equation.

$$y(2x + 1) = 0$$

By the zero product property of real numbers,

$$\begin{aligned} y = 0 \quad &\text{or} \quad 2x + 1 = 0 \\ y = 0 \quad &\text{or} \quad x = -\frac{1}{2}. \end{aligned}$$

If $y = 0$, then the first equation becomes

$$x^2 + x + 1 = 0,$$

which has no real solution. x is assumed to be real, so $y \neq 0$.

On the other hand, if $x = -1/2$, then the first equation becomes

$$\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) + 1 - y^2 = 0$$

$$\frac{3}{4} - y^2 = 0$$

$$y^2 = \frac{3}{4}$$

$$y = \pm\frac{\sqrt{3}}{2}.$$

Therefore, the two solutions to the complex quadratic equation are

$$z = \left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right).$$