Exercise 3

Use established properties of moduli to show that when $|z_3| \neq |z_4|$,

$$\frac{\operatorname{Re}(z_1 + z_2)}{|z_3 + z_4|} \leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}.$$

Solution

Inequality (3) on page 7 states that for a complex number z,

$$\operatorname{Re} z \leq |\operatorname{Re} z| \leq |z|. \tag{3}$$

Inequality (2) on page 11 states that for two complex numbers, z_1 and z_2,

$$|z_1 + z_2| \geq ||z_1| - |z_2||. \tag{2}$$

Use inequality (2) to make the denominator smaller, and use inequality (3) to make the numerator bigger; the fraction becomes bigger as a result in both cases.

$$\begin{aligned} \frac{\operatorname{Re}(z_1 + z_2)}{|z_3 + z_4|} &\leq \frac{|\operatorname{Re}(z_1 + z_2)|}{|z_3 + z_4|} \\ &\leq \frac{|z_1 + z_2|}{|z_3 + z_4|} \\ &\leq \frac{|z_1 + z_2|}{||z_3| - |z_4||} \end{aligned}$$

Therefore, using the triangle inequality ($|z_1| + |z_2| \geq |z_1 + z_2|$) to make the numerator even bigger,

$$\frac{\operatorname{Re}(z_1 + z_2)}{|z_3 + z_4|} \leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}.$$