Exercise 4

Verify that $\sqrt{2}\,|z| \geq |\text{Re}\, z| + |\text{Im}\, z|$.

Suggestion: Reduce this inequality to $(|x| - |y|)^2 \geq 0$.

Solution

Suppose that $z = x + iy$. Then

$$\sqrt{2}\sqrt{x^2 + y^2} \overset{?}{\geq} |x| + |y|$$

$$2(x^2 + y^2) \overset{?}{\geq} (|x| + |y|)^2$$

$$2x^2 + 2y^2 \overset{?}{\geq} x^2 + 2|x||y| + y^2$$

$$x^2 - 2|x||y| + y^2 \overset{?}{\geq} 0$$

$$(|x| - |y|)^2 \geq 0.$$

This inequality is true because a squared quantity is nonnegative, so the inequality in question is verified.