Exercise 14

Using expressions (6), Sec. 6, for $\operatorname{Re} z$ and $\operatorname{Im} z$, show that the hyperbola $x^2 - y^2 = 1$ can be written

$$z^2 + \bar{z}^2 = 2.$$

Solution

x and y are the real and imaginary parts of a complex number $z = x + iy$, respectively. If we have

$$x^2 - y^2 = 1,$$

then

$$(\operatorname{Re} z)^2 - (\operatorname{Im} z)^2 = 1$$

$$\left(\frac{z+\bar{z}}{2}\right)^2 - \left(\frac{z-\bar{z}}{2i}\right)^2 = 1$$

$$\frac{1}{4}(z+\bar{z})^2 - \frac{1}{4i^2}(z-\bar{z})^2 = 1$$

$$\frac{1}{4}(z^2 + 2z\bar{z} + \bar{z}^2) + \frac{1}{4}(z^2 - 2z\bar{z} + \bar{z}^2) = 1.$$

Multiply both sides by 4.

$$(z^2 + 2z\bar{z} + \bar{z}^2) + (z^2 - 2z\bar{z} + \bar{z}^2) = 4$$

$$2z^2 + 2\bar{z}^2 = 4$$

Therefore, the hyperbola $x^2 - y^2 = 1$ can be written as

$$z^2 + \bar{z}^2 = 2.$$