Exercise 15

Follow the steps below to give an algebraic derivation of the triangle inequality (Sec. 5)

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

(*a*) Show that
$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = z_1\overline{z_1} + (z_1\overline{z_2} + \overline{z_1\overline{z_2}}) + z_2\overline{z_2}.$$

(*b*) Point out why
$$z_1\overline{z_2} + \overline{z_1\overline{z_2}} = 2\,\mathrm{Re}(z_1\overline{z_2}) \leq 2|z_1||z_2|.$$

(*c*) Use the results in parts (*a*) and (*b*) to obtain the inequality
$$|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2,$$
and note how the triangle inequality follows.

Solution

Part (a)

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z_1 + z_2}) \\
&= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\
&= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2} \\
&= z_1\overline{z_1} + z_1\overline{z_2} + \overline{z_1}z_2 + z_2\overline{z_2} \\
&= z_1\overline{z_1} + (z_1\overline{z_2} + \overline{z_1\overline{z_2}}) + z_2\overline{z_2}
\end{aligned}$$

Part (b)

$$\begin{aligned}
z_1\overline{z_2} + \overline{z_1\overline{z_2}} &= 2\left(\frac{z_1\overline{z_2} + \overline{z_1\overline{z_2}}}{2}\right) \\
&= 2\,\mathrm{Re}(z_1\overline{z_2}) \\
&\leq 2|z_1\overline{z_2}| \\
&= 2|z_1||\overline{z_2}| \\
&= 2|z_1||z_2|
\end{aligned}$$

Part (c)

$$\begin{aligned}
|z_1 + z_2|^2 &= z_1\overline{z_1} + (z_1\overline{z_2} + \overline{z_1\overline{z_2}}) + z_2\overline{z_2} \\
&\leq z_1\overline{z_1} + (2|z_1||z_2|) + z_2\overline{z_2} \\
&= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \\
&= (|z_1| + |z_2|)^2
\end{aligned}$$

Therefore, taking the square root of both sides,

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$