

## Exercise 1

Use residues to derive the integration formulas in Exercises 1 through 6.

$$\int_0^{\infty} \frac{dx}{x^2 + 1} = \frac{\pi}{2}.$$

### Solution

The integrand is an even function of  $x$ , so the interval of integration can be extended to  $(-\infty, \infty)$  as long as the integral is divided by 2.

$$\int_0^{\infty} \frac{dx}{x^2 + 1} = \int_{-\infty}^{\infty} \frac{dx}{2(x^2 + 1)}$$

In order to evaluate the integral, consider the corresponding function in the complex plane,

$$f(z) = \frac{1}{2(z^2 + 1)},$$

and the contour in Fig. 99. Singularities occur where the denominator is equal to zero.

$$\begin{aligned} 2(z^2 + 1) &= 0 \\ z^2 + 1 &= 0 \\ z &= \pm i \end{aligned}$$

The singular point of interest to us is the one that lies within the closed contour,  $z = i$ .

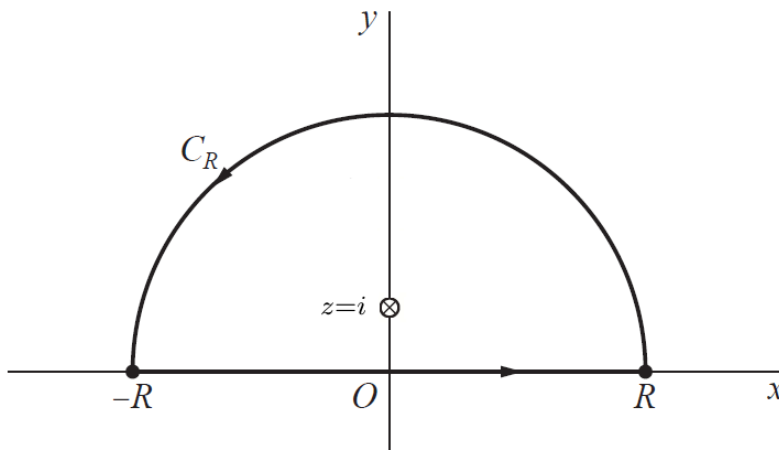


Figure 1: This is Fig. 99 with the singularity at  $z = i$  marked.

According to Cauchy's residue theorem, the integral of  $1/[2(z^2 + 1)]$  around the closed contour is equal to  $2\pi i$  times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{dz}{2(z^2 + 1)} = 2\pi i \operatorname{Res}_{z=i} \frac{1}{2(z^2 + 1)}$$

This closed loop integral is the sum of two integrals, one over each arc in the loop.

$$\int_L \frac{dz}{2(z^2 + 1)} + \int_{C_R} \frac{dz}{2(z^2 + 1)} = 2\pi i \operatorname{Res}_{z=i} \frac{1}{2(z^2 + 1)}$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L: \quad z &= r, & r &= -R \rightarrow r = R \\ C_R: \quad z &= Re^{i\theta}, & \theta &= 0 \rightarrow \theta = \pi \end{aligned}$$

As a result,

$$\int_{-R}^R \frac{dr}{2(r^2 + 1)} + \int_{C_R} \frac{dz}{2(z^2 + 1)} = 2\pi i \operatorname{Res}_{z=i} \frac{1}{2(z^2 + 1)}.$$

Take the limit now as  $R \rightarrow \infty$ . The integral over  $C_R$  consequently tends to zero. Proof for this statement will be given at the end.

$$\int_{-\infty}^{\infty} \frac{dr}{2(r^2 + 1)} = 2\pi i \operatorname{Res}_{z=i} \frac{1}{2(z^2 + 1)}$$

The denominator can be written as  $2(z^2 + 1) = 2(z + i)(z - i)$ . From this we see that the multiplicity of the  $z - i$  factor is 1. The residue at  $z = i$  can then be calculated by

$$\operatorname{Res}_{z=i} \frac{1}{2(z^2 + 1)} = \phi(i),$$

where  $\phi(z)$  is equal to  $f(z)$  without the  $z - i$  factor.

$$\phi(z) = \frac{1}{2(z + i)} \Rightarrow \phi(i) = \frac{1}{4i}$$

So then

$$\operatorname{Res}_{z=i} \frac{1}{2(z^2 + 1)} = \frac{1}{4i}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dr}{2(r^2 + 1)} &= 2\pi i \left( \frac{1}{4i} \right) \\ &= \frac{\pi}{2}. \end{aligned}$$

Therefore, changing the dummy integration variable to  $x$ ,

$$\boxed{\int_0^{\infty} \frac{dx}{x^2 + 1} = \frac{\pi}{2}.}$$

The Integral Over  $C_R$ 

Our aim here is to show that the integral over  $C_R$  tends to zero in the limit as  $R \rightarrow \infty$ . The parameterization of the semicircular arc in Fig. 99 is  $z = Re^{i\theta}$ , where  $\theta$  goes from 0 to  $\pi$ .

$$\begin{aligned} \int_{C_R} \frac{dz}{2(z^2 + 1)} &= \int_0^\pi \frac{Rie^{i\theta} d\theta}{2[(Re^{i\theta})^2 + 1]} \\ &= \int_0^\pi \frac{Rie^{i\theta} d\theta}{2(R^2e^{i2\theta} + 1)} \end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned} \left| \int_{C_R} \frac{dz}{2(z^2 + 1)} \right| &= \left| \int_0^\pi \frac{Rie^{i\theta} d\theta}{2(R^2e^{i2\theta} + 1)} \right| \\ &\leq \int_0^\pi \left| \frac{Rie^{i\theta}}{2(R^2e^{i2\theta} + 1)} \right| d\theta \\ &= \int_0^\pi \frac{|Rie^{i\theta}|}{|2(R^2e^{i2\theta} + 1)|} d\theta \\ &= \int_0^\pi \frac{R}{|R^2e^{i2\theta} + 1|} \frac{d\theta}{2} \\ &\leq \int_0^\pi \frac{R}{|R^2e^{i2\theta}| - |1|} \frac{d\theta}{2} \\ &= \int_0^\pi \frac{R}{R^2 - 1} \frac{d\theta}{2} \\ &= \frac{\pi}{2} \frac{R}{R^2 - 1} \end{aligned}$$

Now take the limit of both sides as  $R \rightarrow \infty$ .

$$\begin{aligned} \lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{dz}{2(z^2 + 1)} \right| &\leq \lim_{R \rightarrow \infty} \frac{\pi}{2} \frac{R}{R^2 - 1} \\ &= \lim_{R \rightarrow \infty} \frac{\pi}{2R} \frac{1}{1 - \frac{1}{R^2}} \end{aligned}$$

The limit on the right side is zero.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{dz}{2(z^2 + 1)} \right| \leq 0$$

The magnitude of a number cannot be negative.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{dz}{2(z^2 + 1)} dz \right| = 0$$

The only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R \rightarrow \infty} \int_{C_R} \frac{dz}{2(z^2 + 1)} dz = 0.$$