Exercise 10

Let m and n be integers, where $0 \leq m < n$. Follow the steps below to derive the integration formula

$$\int_0^\infty \frac{x^{2m}}{x^{2n}+1}\,dx = \frac{\pi}{2n}\csc\left(\frac{2m+1}{2n}\pi\right).$$

(a) Show that the zeros of the polynomial $z^{2n}+1$ lying above the real axis are

$$c_k = \exp\left[i\frac{(2k+1)\pi}{2n}\right] \qquad (k = 0, 1, 2, \ldots, n-1)$$

and that there are none on that axis.

(b) With the aid of Theorem 2 in Sec. 83, show that

$$\operatorname*{Res}_{z=c_k} \frac{z^{2m}}{z^{2n}+1} = -\frac{1}{2n}e^{i(2k+1)\alpha} \qquad (k = 0, 1, 2, \ldots, n-1)$$

where c_k are the zeros found in part (a) and

$$\alpha = \frac{2m+1}{2n}\pi.$$

Then use the summation formula

$$\sum_{k=0}^{n-1} z^k = \frac{1-z^n}{1-z} \qquad (z \neq 1)$$

(see Exercise 9, Sec. 9) to obtain the expression

$$2\pi i \sum_{k=0}^{n-1} \operatorname*{Res}_{z=c_k} \frac{z^{2m}}{z^{2n}+1} = \frac{\pi}{n\sin\alpha}.$$

(c) Use the final result in part (b) to complete the derivation of the integration formula.

Solution

The integrand is an even function of x, so the interval of integration can be extended to $(-\infty, \infty)$ as long as the integral is divided by 2.

$$\int_0^\infty \frac{x^{2m}}{x^{2n}+1}\,dx = \int_0^\infty \frac{(x^2)^m}{(x^2)^n+1}\,dx = \int_{-\infty}^\infty \frac{x^{2m}}{2(x^{2n}+1)}\,dx$$

In order to evaluate the integral, consider the corresponding function in the complex plane,

$$f(z) = \frac{z^{2m}}{2(z^{2n}+1)},$$

and the contour in Fig. 99. Singularities occur where the denominator is equal to zero.

$$2(z^{2n}+1) = 0$$

$$z^{2n}+1 = 0$$

$$z = \sqrt[2n]{1}\exp\left[i\left(\frac{\pi+2k\pi}{2n}\right)\right], \quad k = 0, 1, \ldots, 2n-1$$

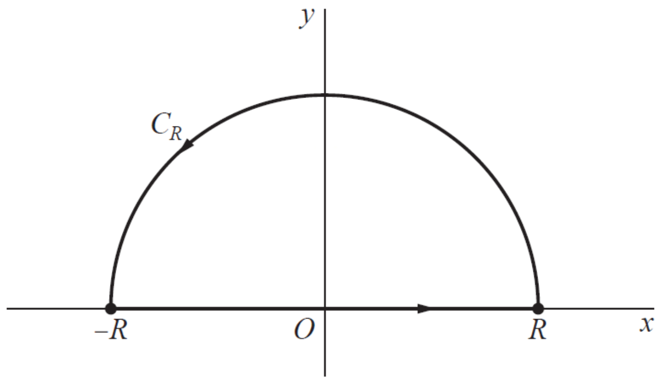

Figure 1: This is Fig. 99.

The singular points of interest to us are the ones that lie within the closed contour, that is, those with a positive imaginary component. Use Euler's formula to write the exponential function in terms of sine and cosine.

$$z = \cos\left(\frac{\pi + 2k\pi}{2n}\right) + i\sin\left(\frac{\pi + 2k\pi}{2n}\right), \quad k = 0, 1, \ldots, 2n-1$$

We require

$$\sin\left(\frac{\pi + 2k\pi}{2n}\right) > 0, \quad k = 0, 1, \ldots, 2n-1,$$

so the argument of sine must have a value between 0 and π.

$$0 < \frac{\pi + 2k\pi}{2n} < \pi$$
$$0 < \frac{1 + 2k}{2n} < 1$$
$$0 < 1 + 2k < 2n$$

The values of k that satisfy this inequality are $k = 0, 1, \ldots n-1$. Thus, the singular points that lie within the contour are

$$z = z_k = \exp\left[i\left(\frac{\pi + 2k\pi}{2n}\right)\right], \quad k = 0, 1, \ldots, n-1.$$

According to Cauchy's residue theorem, the integral of $z^{2m}/[2(z^{2n}+1)]$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{z^{2m}}{2(z^{2n}+1)}\,dz = 2\pi i \sum_{k=0}^{n-1} \operatorname*{Res}_{z=z_k} \frac{z^{2m}}{2(z^{2n}+1)}$$

This closed loop integral is the sum of two integrals, one over each arc in the loop.

$$\int_L \frac{z^{2m}}{2(z^{2n}+1)}\,dz + \int_{C_R} \frac{z^{2m}}{2(z^{2n}+1)}\,dz = 2\pi i \sum_{k=0}^{n-1} \operatorname*{Res}_{z=z_k} \frac{z^{2m}}{2(z^{2n}+1)}$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L: &\quad z = r, &\quad r = -R \quad &\to \quad r = R \\ C_R: &\quad z = Re^{i\theta}, &\quad \theta = 0 \quad &\to \quad \theta = \pi \end{aligned}$$

As a result,

$$\int_{-R}^{R} \frac{r^{2m}}{2(r^{2n}+1)}\,dr + \int_{C_R} \frac{z^{2m}}{2(z^{2n}+1)}\,dz = 2\pi i \sum_{k=0}^{n-1} \operatorname*{Res}_{z=z_k} \frac{z^{2m}}{2(z^{2n}+1)}.$$

Take the limit now as $R \to \infty$. The integral over C_R consequently tends to zero. Proof for this statement will be given at the end.

$$\int_{-\infty}^{\infty} \frac{r^{2m}}{2(r^{2n}+1)}\,dr = 2\pi i \sum_{k=0}^{n-1} \operatorname*{Res}_{z=z_k} \frac{z^{2m}}{2(z^{2n}+1)}$$

The residue at $z = z_k$ can be calculated by

$$\operatorname*{Res}_{z=z_k} \frac{z^{2m}}{2(z^{2n}+1)} = \frac{p(z_k)}{q'(z_k)},$$

where $p(z)$ and $q(z)$ are equal to the numerator and denominator of $f(z)$, respectively.

$$p(z) = z^{2m} \quad \Rightarrow \quad p(z_k) = \exp\left[i\left(\frac{\pi+2k\pi}{2n}\right)2m\right]$$

$$q(z) = 2(z^{2n}+1) \quad \rightarrow \quad q'(z) = 4nz^{2n-1} \quad \Rightarrow \quad q'(z_k) = 4n\exp\left[i\left(\frac{\pi+2k\pi}{2n}\right)(2n-1)\right]$$

So then

$$\begin{aligned}
\operatorname*{Res}_{z=z_k} \frac{z^{2m}}{2(z^{2n}+1)} &= \frac{\exp\left[i\left(\frac{\pi+2k\pi}{2n}\right)2m\right]}{4n\exp\left[i\left(\frac{\pi+2k\pi}{2n}\right)(2n-1)\right]} \\
&= \frac{1}{4n}\exp\left[i\left(\frac{\pi+2k\pi}{2n}\right)(2m-2n+1)\right] \\
&= \frac{1}{4n}\exp\left[i(2k+1)\frac{2m-2n+1}{2n}\pi\right] \\
&= \frac{1}{4n}\exp\left[i(2k+1)\frac{2m+1}{2n}\pi\right]\exp\left[i(2k+1)(-1)\pi\right] \\
&= \frac{1}{4n}\exp\left[i(2k+1)\frac{2m+1}{2n}\pi\right](-1) \\
&= -\frac{1}{4n}\exp\left(ik\frac{2m+1}{n}\pi\right)\exp\left(i\frac{2m+1}{2n}\pi\right)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{n-1} \operatorname*{Res}_{z=z_k} \frac{z^{2m}}{2(z^{2n}+1)} &= \sum_{k=0}^{n-1}\left\{-\frac{1}{4n}\exp\left(ik\frac{2m+1}{n}\pi\right)\exp\left(i\frac{2m+1}{2n}\pi\right)\right\} \\
&= -\frac{1}{4n}\exp\left(i\frac{2m+1}{2n}\pi\right)\sum_{k=0}^{n-1}\exp\left(ik\frac{2m+1}{n}\pi\right) \\
&= -\frac{1}{4n}\exp\left(i\frac{2m+1}{2n}\pi\right)\sum_{k=0}^{n-1}\left[\exp\left(i\frac{2m+1}{n}\pi\right)\right]^k \\
&= -\frac{1}{4n}\exp\left(i\frac{2m+1}{2n}\pi\right)\frac{1-\left[\exp\left(i\frac{2m+1}{n}\pi\right)\right]^n}{1-\exp\left(i\frac{2m+1}{n}\pi\right)}
\end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{n-1} \operatorname*{Res}_{z=z_k} \frac{z^{2m}}{2(z^{2n}+1)} &= -\frac{1}{4n} \exp\left(i\frac{2m+1}{2n}\pi\right) \frac{1-\exp[i(2m+1)\pi]}{1-\exp\left(i\frac{2m+1}{n}\pi\right)} \\
&= -\frac{1}{4n} \exp\left(i\frac{2m+1}{2n}\pi\right) \frac{1-(-1)}{1-\exp\left(i\frac{2m+1}{n}\pi\right)} \\
&= -\frac{1}{2n} \exp\left(i\frac{2m+1}{2n}\pi\right) \frac{1}{1-\exp\left(i\frac{2m+1}{n}\pi\right)} \\
&= -\frac{1}{2n} \left[\frac{1}{\exp\left(-i\frac{2m+1}{2n}\pi\right)-\exp\left(i\frac{2m+1}{2n}\pi\right)}\right] \\
&= -\frac{1}{2n} \left[\frac{1}{-2i\sin\left(\frac{2m+1}{2n}\pi\right)}\right] \\
&= \frac{1}{2n} \frac{1}{2i\sin\left(\frac{2m+1}{2n}\pi\right)}
\end{aligned}$$

and

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{r^{2m}}{2(r^{2n}+1)}\,dr &= 2\pi i \left[\frac{1}{2n} \frac{1}{2i\sin\left(\frac{2m+1}{2n}\pi\right)}\right] \\
&= \frac{\pi}{2n} \frac{1}{\sin\left(\frac{2m+1}{2n}\pi\right)} \\
&= \frac{\pi}{2n} \csc\left(\frac{2m+1}{2n}\pi\right).
\end{aligned}$$

Therefore, changing the dummy integration variable to x,

$$\boxed{\int_0^{\infty} \frac{x^{2m}}{x^{2n}+1}\,dx = \frac{\pi}{2n} \csc\left(\frac{2m+1}{2n}\pi\right).}$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \to \infty$. The parameterization of the semicircular arc in Fig. 99 is $z = Re^{i\theta}$, where θ goes from 0 to π.

$$\begin{aligned}\int_{C_R} \frac{z^{2m}}{2(z^{2n}+1)}\,dz &= \int_0^\pi \frac{(Re^{i\theta})^{2m}}{2[(Re^{i\theta})^{2n}+1]}(Rie^{i\theta}\,d\theta) \\ &= \int_0^\pi \frac{R^{2m+1}ie^{i\theta(2m+1)}}{R^{2n}e^{i2n\theta}+1}\,\frac{d\theta}{2}\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}\left|\int_{C_R} \frac{z^{2m}}{2(z^{2n}+1)}\,dz\right| &= \left|\int_0^\pi \frac{R^{2m+1}ie^{i\theta(2m+1)}}{R^{2n}e^{i2n\theta}+1}\,\frac{d\theta}{2}\right| \\ &\leq \int_0^\pi \left|\frac{R^{2m+1}ie^{i\theta(2m+1)}}{R^{2n}e^{i2n\theta}+1}\right|\frac{d\theta}{2} \\ &= \int_0^\pi \frac{\left|R^{2m+1}ie^{i\theta(2m+1)}\right|}{|R^{2n}e^{i2n\theta}+1|}\,\frac{d\theta}{2} \\ &= \int_0^\pi \frac{R^{2m+1}}{|R^{2n}e^{i2n\theta}+1|}\,\frac{d\theta}{2} \\ &\leq \int_0^\pi \frac{R^{2m+1}}{|R^{2n}e^{i2n\theta}|-|1|}\,\frac{d\theta}{2} \\ &= \int_0^\pi \frac{R^{2m+1}}{R^{2n}-1}\,\frac{d\theta}{2} \\ &= \frac{\pi}{2}\frac{R^{2m+1}}{R^{2n}-1}\end{aligned}$$

Now take the limit of both sides as $R \to \infty$.

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{C_R} \frac{z^{2m}}{2(z^{2n}+1)}\,dz\right| &\leq \lim_{R\to\infty}\frac{\pi}{2}\frac{R^{2m+1}}{R^{2n}-1} \\ &= \lim_{R\to\infty}\frac{\pi}{2R^{2n-2m-1}}\frac{1}{1-\frac{1}{R^{2n}}}\end{aligned}$$

Since $n > m$ and n and m are integers, $2n - 2m - 1 > 0$, and the limit on the right side is zero.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{z^{2m}}{2(z^{2n}+1)}\,dz\right| \leq 0$$

The magnitude of a number cannot be negative.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{z^{2m}}{2(z^{2n}+1)}\,dz\right| = 0$$

The only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R\to\infty}\int_{C_R} \frac{z^{2m}}{2(z^{2n}+1)}\,dz = 0.$$