

Exercise 6

Use residues to derive the integration formulas in Exercises 1 through 6.

$$\int_0^{\infty} \frac{x^2 dx}{(x^2 + 9)(x^2 + 4)^2} = \frac{\pi}{200}.$$

Solution

The integrand is an even function of x , so the interval of integration can be extended to $(-\infty, \infty)$ as long as the integral is divided by 2.

$$\int_0^{\infty} \frac{x^2 dx}{(x^2 + 9)(x^2 + 4)^2} = \int_{-\infty}^{\infty} \frac{x^2 dx}{2(x^2 + 9)(x^2 + 4)^2}$$

In order to evaluate the integral, consider the corresponding function in the complex plane,

$$f(z) = \frac{z^2}{2(z^2 + 9)(z^2 + 4)^2},$$

and the contour in Fig. 99. Singularities occur where the denominator is equal to zero.

$$\begin{aligned} 2(z^2 + 9)(z^2 + 4)^2 &= 0 \\ z^2 + 9 = 0 &\quad \text{or} \quad z^2 + 4 = 0 \\ z = \pm 3i &\quad \text{or} \quad z = \pm 2i \end{aligned}$$

The singular points of interest to us are the ones that lie within the closed contour, $z = 2i$ and $z = 3i$.

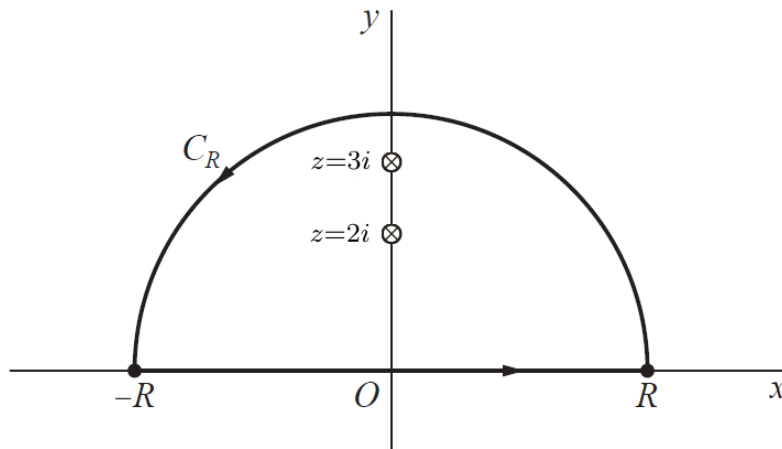


Figure 1: This is Fig. 99 with the singularities at $z = 2i$ and $z = 3i$ marked.

According to Cauchy's residue theorem, the integral of $z^2/[2(z^2 + 9)(z^2 + 4)^2]$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{z^2 dz}{2(z^2 + 9)(z^2 + 4)^2} = 2\pi i \left[\operatorname{Res}_{z=2i} \frac{z^2}{2(z^2 + 9)(z^2 + 4)^2} + \operatorname{Res}_{z=3i} \frac{z^2}{2(z^2 + 9)(z^2 + 4)^2} \right]$$

This closed loop integral is the sum of two integrals, one over each arc in the loop.

$$\begin{aligned} \int_L \frac{z^2 dz}{2(z^2 + 9)(z^2 + 4)^2} + \int_{C_R} \frac{z^2 dz}{2(z^2 + 9)(z^2 + 4)^2} \\ = 2\pi i \left[\operatorname{Res}_{z=2i} \frac{z^2}{2(z^2 + 9)(z^2 + 4)^2} + \operatorname{Res}_{z=3i} \frac{z^2}{2(z^2 + 9)(z^2 + 4)^2} \right] \end{aligned}$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L: \quad z = r, \quad r = -R \rightarrow r = R \\ C_R: \quad z = Re^{i\theta}, \quad \theta = 0 \rightarrow \theta = \pi \end{aligned}$$

As a result,

$$\begin{aligned} \int_{-R}^R \frac{r^2 dr}{2(r^2 + 9)(r^2 + 4)^2} + \int_{C_R} \frac{z^2 dz}{2(z^2 + 9)(z^2 + 4)^2} \\ = 2\pi i \left[\operatorname{Res}_{z=2i} \frac{z^2}{2(z^2 + 9)(z^2 + 4)^2} + \operatorname{Res}_{z=3i} \frac{z^2}{2(z^2 + 9)(z^2 + 4)^2} \right]. \end{aligned}$$

Take the limit now as $R \rightarrow \infty$. The integral over C_R consequently tends to zero. Proof for this statement will be given at the end.

$$\int_{-\infty}^{\infty} \frac{r^2 dr}{2(r^2 + 9)(r^2 + 4)^2} = 2\pi i \left[\operatorname{Res}_{z=2i} \frac{z^2}{2(z^2 + 9)(z^2 + 4)^2} + \operatorname{Res}_{z=3i} \frac{z^2}{2(z^2 + 9)(z^2 + 4)^2} \right]$$

The denominator can be written as $2(z^2 + 9)(z^2 + 4)^2 = 2(z + 3i)(z - 3i)(z + 2i)^2(z - 2i)^2$. From this we see that the multiplicities of the $z - 2i$ and $z - 3i$ factors are 2 and 1, respectively. The residues at $z = 2i$ and $z = 3i$ can then be calculated by

$$\begin{aligned} \operatorname{Res}_{z=2i} \frac{z^2}{2(z^2 + 9)(z^2 + 4)^2} &= \frac{\phi_1^{(2-1)}(2i)}{(2-1)!} = \phi_1'(2i) \\ \operatorname{Res}_{z=3i} \frac{z^2}{2(z^2 + 9)(z^2 + 4)^2} &= \phi_2(3i), \end{aligned}$$

where $\phi_1(z)$ and $\phi_2(z)$ are equal to $f(z)$ without the $(z - 2i)^2$ and $z - 3i$ factors, respectively.

$$\begin{aligned} \phi_1(z) = \frac{z^2}{2(z + 3i)(z - 3i)(z + 2i)^2} \quad \rightarrow \quad \phi_1'(z) = -\frac{z(z^3 - 18i)}{(z + 2i)^3(z^2 + 9)^2} \quad \Rightarrow \quad \phi_1'(2i) = -\frac{13i}{400} \\ \phi_2(z) = \frac{z^2}{2(z + 3i)(z + 2i)^2(z - 2i)^2} \quad \Rightarrow \quad \phi_2(3i) = \frac{3i}{100} \end{aligned}$$

So then

$$\begin{aligned} \operatorname{Res}_{z=2i} \frac{z^2}{2(z^2 + 9)(z^2 + 4)^2} &= -\frac{13i}{400} \\ \operatorname{Res}_{z=3i} \frac{z^2}{2(z^2 + 9)(z^2 + 4)^2} &= \frac{3i}{100} \end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{r^2 dr}{2(r^2 + 9)(r^2 + 4)^2} &= 2\pi i \left(-\frac{13i}{400} + \frac{3i}{100} \right) \\ &= 2\pi i \left(-\frac{i}{400} \right) \\ &= \frac{\pi}{200}.\end{aligned}$$

Therefore, changing the dummy integration variable to x ,

$$\boxed{\int_0^{\infty} \frac{x^2 dx}{(x^2 + 9)(x^2 + 4)^2} = \frac{\pi}{200}.$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \rightarrow \infty$. The parameterization of the semicircular arc in Fig. 99 is $z = Re^{i\theta}$, where θ goes from 0 to π .

$$\begin{aligned} \int_{C_R} \frac{z^2 dz}{2(z^2 + 9)(z^2 + 4)^2} &= \int_0^\pi \frac{(Re^{i\theta})^2 (Rie^{i\theta} d\theta)}{2[(Re^{i\theta})^2 + 9][(Re^{i\theta})^2 + 4]^2} \\ &= \int_0^\pi \frac{R^3 ie^{i3\theta}}{(R^2 e^{i2\theta} + 9)(R^2 e^{i2\theta} + 4)^2} \frac{d\theta}{2} \end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned} \left| \int_{C_R} \frac{z^2 dz}{2(z^2 + 9)(z^2 + 4)^2} \right| &= \left| \int_0^\pi \frac{R^3 ie^{i3\theta}}{(R^2 e^{i2\theta} + 9)(R^2 e^{i2\theta} + 4)^2} \frac{d\theta}{2} \right| \\ &\leq \int_0^\pi \left| \frac{R^3 ie^{i3\theta}}{(R^2 e^{i2\theta} + 9)(R^2 e^{i2\theta} + 4)^2} \right| \frac{d\theta}{2} \\ &= \int_0^\pi \frac{|R^3 ie^{i3\theta}|}{|R^2 e^{i2\theta} + 9| |R^2 e^{i2\theta} + 4|^2} \frac{d\theta}{2} \\ &= \int_0^\pi \frac{R^3}{|R^2 e^{i2\theta} + 9| |R^2 e^{i2\theta} + 4|^2} \frac{d\theta}{2} \\ &\leq \int_0^\pi \frac{R^3}{(|R^2 e^{i2\theta}| - |9|)(|R^2 e^{i2\theta}| - |4|)^2} \frac{d\theta}{2} \\ &= \int_0^\pi \frac{R^3}{(R^2 - 9)(R^2 - 4)^2} \frac{d\theta}{2} \\ &= \frac{\pi}{2} \frac{R^3}{(R^2 - 9)(R^2 - 4)^2} \end{aligned}$$

Now take the limit of both sides as $R \rightarrow \infty$.

$$\begin{aligned} \lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{z^2 dz}{2(z^2 + 9)(z^2 + 4)^2} \right| &\leq \lim_{R \rightarrow \infty} \frac{\pi}{2} \frac{R^3}{(R^2 - 9)(R^2 - 4)^2} \\ &= \lim_{R \rightarrow \infty} \frac{\pi}{2R^3} \frac{1}{\left(1 - \frac{9}{R^2}\right) \left(1 - \frac{4}{R^2}\right)^2} \end{aligned}$$

The limit on the right side is zero.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{z^2 dz}{2(z^2 + 9)(z^2 + 4)^2} \right| \leq 0$$

The magnitude of a number cannot be negative.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{z^2 dz}{2(z^2 + 9)(z^2 + 4)^2} \right| = 0$$

The only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R \rightarrow \infty} \int_{C_R} \frac{z^2 dz}{2(z^2 + 9)(z^2 + 4)^2} = 0.$$