# Exercise 12

Follow the steps below to evaluate the ***Fresnel integrals***, which are important in diffraction theory:

$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

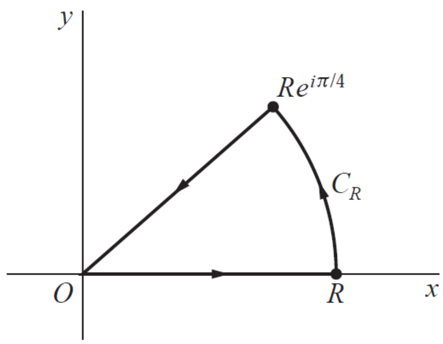


FIGURE 106

(*a*) By integrating the function $\exp(iz^2)$ around the positively oriented boundary of the sector $0 \le r \le R$, $0 \le \theta \le \pi/4$ (Fig. 106) and appealing to the Cauchy-Goursat theorem, show that

$$\int_0^R \cos(x^2)\,dx = \frac{1}{\sqrt{2}}\int_0^R e^{-r^2}\,dr - \operatorname{Re}\int_{C_R} e^{iz^2}\,dz$$

and

$$\int_0^R \sin(x^2)\,dx = \frac{1}{\sqrt{2}}\int_0^R e^{-r^2}\,dr - \operatorname{Im}\int_{C_R} e^{iz^2}\,dz,$$

where $C_R$ is the arc $z = Re^{i\theta}$ $(0 \le \theta \le \pi/4)$.

(*b*) Show that the value of the integral along the arc $C_R$ in part (*a*) tends to zero as $R$ tends to infinity by obtaining the inequality

$$\left|\int_{C_R} e^{iz^2}\,dz\right| \le \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin\phi}\,d\phi$$

and then referring to the form (2), Sec. 88, of Jordan's inequality.

(*c*) Use the results in parts (*a*) and (*b*), together with the known integration formula*

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2},$$

to complete the exercise.

*The usual way to evaluate this integral is by writing its square as

$$\int_0^\infty e^{-x^2}\,dx \int_0^\infty e^{-y^2}\,dy = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

and then evaluating this iterated integral by changing to polar coordinates. Details are given in, for example, A. E. Taylor and W. R. Mann, "Advanced Calculus," 3d ed., pp. 680–681, 1983.

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**Solution**

**Part (a)**

In order to evaluate the Fresnel integrals, consider the corresponding function in the complex plane,

$$f(z) = e^{iz^2},$$

and the contour in Fig. 106. According to the Cauchy-Goursat theorem, the integral of $e^{iz^2}$ around the closed contour is equal to zero because the function has no singularities within it.

$$\oint_C e^{iz^2}\,dz = 0$$

This closed loop integral is the sum of three integrals, one over each arc in the loop.

$$\int_{L_1} e^{iz^2}\,dz + \int_{L_2} e^{iz^2}\,dz + \int_{C_R} e^{iz^2}\,dz = 0$$

The parameterizations for the arcs are as follows.

$$\begin{aligned}
L_1: &\quad z = re^{i0}, && r = 0 \quad \to \quad r = R \\
L_2: &\quad z = re^{i\pi/4}, && r = R \quad \to \quad r = 0 \\
C_R: &\quad z = Re^{i\theta}, && \theta = 0 \quad \to \quad \theta = \frac{\pi}{4}
\end{aligned}$$

As a result,

$$\begin{aligned}
0 &= \int_0^R e^{i(re^{i0})^2}(dr\,e^{i0}) + \int_R^0 e^{i(re^{i\pi/4})^2}(dr\,e^{i\pi/4}) + \int_{C_R} e^{iz^2}\,dz \\
0 &= \int_0^R e^{ir^2}\,dr + e^{i\pi/4}\int_R^0 e^{ir^2e^{i\pi/2}}\,dr + \int_{C_R} e^{iz^2}\,dz \\
0 &= \int_0^R (\cos r^2 + i\sin r^2)\,dr + \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\int_R^0 e^{i^2r^2}\,dr + \int_{C_R} e^{iz^2}\,dz \\
0 &= \int_0^R (\cos r^2 + i\sin r^2)\,dr - \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\int_0^R e^{-r^2}\,dr + \int_{C_R} e^{iz^2}\,dz \\
0 &= \int_0^R \cos r^2\,dr + i\int_0^R \sin r^2\,dr - \frac{1}{\sqrt{2}}\int_0^R e^{-r^2}\,dr - i\frac{1}{\sqrt{2}}\int_0^R e^{-r^2}\,dr + \int_{C_R} e^{iz^2}\,dz.
\end{aligned}$$

Match the real and imaginary parts of both sides of the equation to obtain two separate equations, one involving the integral of cosine and one involving the integral of sine.

$$\begin{aligned}
\int_0^R \cos r^2\,dr - \frac{1}{\sqrt{2}}\int_0^R e^{-r^2}\,dr + \operatorname{Re}\int_{C_R} e^{iz^2}\,dz = 0 \\
\int_0^R \sin r^2\,dr - \frac{1}{\sqrt{2}}\int_0^R e^{-r^2}\,dr + \operatorname{Im}\int_{C_R} e^{iz^2}\,dz = 0
\end{aligned}$$

Solve both equations for the desired integrals.

$$\int_0^R \cos r^2\,dr = \frac{1}{\sqrt{2}}\int_0^R e^{-r^2}\,dr - \operatorname{Re}\int_{C_R} e^{iz^2}\,dz$$
$$\int_0^R \sin r^2\,dr = \frac{1}{\sqrt{2}}\int_0^R e^{-r^2}\,dr - \operatorname{Im}\int_{C_R} e^{iz^2}\,dz.$$

## Part (b)

Here we will show that the integral over $C_R$ tends to zero as $R \to \infty$. The parameterization of $C_R$ is $z = Re^{i\theta}$, where $\theta$ goes from 0 to $\pi/4$.

$$\begin{aligned}
\int_{C_R} e^{iz^2}\,dz &= \int_0^{\pi/4} e^{i(Re^{i\theta})^2}(Rie^{i\theta}\,d\theta) \\
&= \int_0^{\pi/4} e^{iR^2e^{2i\theta}}(Rie^{i\theta}\,d\theta) \\
&= \int_0^{\pi/4} e^{iR^2(\cos 2\theta + i\sin 2\theta)}(Rie^{i\theta}\,d\theta) \\
&= \int_0^{\pi/4} e^{iR^2\cos 2\theta}e^{-R^2\sin 2\theta}(Rie^{i\theta}\,d\theta)
\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}
\left|\int_{C_R} e^{iz^2}\,dz\right| &= \left|\int_0^{\pi/4} e^{iR^2\cos 2\theta}e^{-R^2\sin 2\theta}(Rie^{i\theta}\,d\theta)\right| \\
&\le \int_0^{\pi/4} \left|e^{iR^2\cos 2\theta}e^{-R^2\sin 2\theta}(Rie^{i\theta})\right|d\theta \\
&= \int_0^{\pi/4} \left|e^{iR^2\cos 2\theta}\right|\left|e^{-R^2\sin 2\theta}\right|\left|Rie^{i\theta}\right|d\theta \\
&= \int_0^{\pi/4} e^{-R^2\sin 2\theta}R\,d\theta
\end{aligned}$$

Take the limit now as $R \to \infty$.

$$\lim_{R\to\infty}\left|\int_{C_R} e^{iz^2}\,dz\right| \le \lim_{R\to\infty}\int_0^{\pi/4} e^{-R^2\sin 2\theta}R\,d\theta$$

Because the limits of integration are constant, the limit may be brought inside the integral.

$$= \int_0^{\pi/4} \lim_{R\to\infty}\frac{R}{e^{R^2\sin 2\theta}}\,d\theta$$

The indeterminate form $\infty/\infty$ is obtained as $R \to \infty$, so l'Hôpital's rule will be applied to calculate the limit.

$$\overset{\frac{\infty}{\infty}}{\underset{\text{H}}{=}} \int_0^{\pi/4} \lim_{R\to\infty}\frac{1}{e^{R^2\sin 2\theta}\cdot 2R\sin 2\theta}\,d\theta$$

Since $\theta$ lies between 0 and $\pi/4$, the sine of $2\theta$ is positive, and the denominator goes to infinity. Consequently,

$$\lim_{R\to\infty}\left|\int_{C_R} e^{iz^2}\,dz\right| \le 0.$$

The magnitude of a number cannot be negative.

$$\lim_{R\to\infty}\left|\int_{C_R} e^{iz^2}\,dz\right| = 0$$

The only number that has a magnitude of zero is zero.

$$\lim_{R\to\infty} \int_{C_R} e^{iz^2}\, dz = 0$$

Matching the real and imaginary parts of both sides of this equation,

$$\lim_{R\to\infty} \operatorname{Re} \int_{C_R} e^{iz^2}\, dz = 0 \quad \text{and} \quad \lim_{R\to\infty} \operatorname{Im} \int_{C_R} e^{iz^2}\, dz = 0.$$

**Part (c)**

In the limit as $R \to \infty$ the result of part (a) becomes

$$\begin{aligned}
\int_0^\infty \cos r^2\, dr &= \frac{1}{\sqrt{2}} \int_0^\infty e^{-r^2}\, dr - \overbrace{\lim_{R\to\infty} \operatorname{Re} \int_{C_R} e^{iz^2}\, dz}^{=0} = \frac{1}{\sqrt{2}} \frac{\sqrt{\pi}}{2} \\
\int_0^\infty \sin r^2\, dr &= \frac{1}{\sqrt{2}} \int_0^\infty e^{-r^2}\, dr - \underbrace{\lim_{R\to\infty} \operatorname{Im} \int_{C_R} e^{iz^2}\, dz}_{=0} = \frac{1}{\sqrt{2}} \frac{\sqrt{\pi}}{2}.
\end{aligned}$$

Therefore, changing the dummy integration variable to $x$,

$$\int_0^\infty \cos(x^2)\, dx = \int_0^\infty \sin(x^2)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$