

Exercise 6

Use residues to evaluate the integrals in Exercises 6 and 7.

$$\int_{-\infty}^{\infty} \frac{x \sin x \, dx}{(x^2 + 1)(x^2 + 4)}.$$

Solution

In order to evaluate the integral, consider the corresponding function in the complex plane,

$$f(z) = \frac{ze^{iz}}{(z^2 + 1)(z^2 + 4)},$$

and the contour in Fig. 99. Singularities occur where the denominator is equal to zero.

$$\begin{aligned} (z^2 + 1)(z^2 + 4) &= 0 \\ z^2 + 1 = 0 \quad \text{or} \quad z^2 + 4 &= 0 \\ z = \pm i \quad \text{or} \quad z = \pm 2i \end{aligned}$$

The singular points of interest to us are the ones that lie within the closed contour, $z = i$ and $z = 2i$.

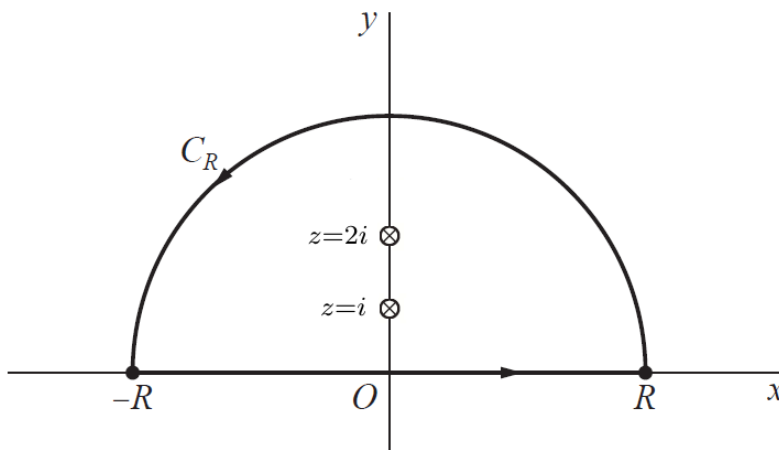


Figure 1: This is Fig. 99 with the singularities at $z = i$ and $z = 2i$ marked.

According to Cauchy's residue theorem, the integral of $ze^{iz}/[(z^2 + 1)(z^2 + 4)]$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{ze^{iz}}{(z^2 + 1)(z^2 + 4)} dz = 2\pi i \left[\operatorname{Res}_{z=i} \frac{ze^{iz}}{(z^2 + 1)(z^2 + 4)} + \operatorname{Res}_{z=2i} \frac{ze^{iz}}{(z^2 + 1)(z^2 + 4)} \right]$$

This closed loop integral is the sum of two integrals, one over each arc in the loop.

$$\begin{aligned} \int_L \frac{ze^{iz}}{(z^2 + 1)(z^2 + 4)} dz + \int_{C_R} \frac{ze^{iz}}{(z^2 + 1)(z^2 + 4)} dz \\ = 2\pi i \left[\operatorname{Res}_{z=i} \frac{ze^{iz}}{(z^2 + 1)(z^2 + 4)} + \operatorname{Res}_{z=2i} \frac{ze^{iz}}{(z^2 + 1)(z^2 + 4)} \right] \end{aligned}$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L: \quad z &= r, & r &= -R \rightarrow r = R \\ C_R: \quad z &= Re^{i\theta}, & \theta &= 0 \rightarrow \theta = \pi \end{aligned}$$

As a result,

$$\begin{aligned} \int_{-R}^R \frac{re^{ir}}{(r^2+1)(r^2+4)} dr + \int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)} dz \\ = 2\pi i \left[\operatorname{Res}_{z=i} \frac{ze^{iz}}{(z^2+1)(z^2+4)} + \operatorname{Res}_{z=2i} \frac{ze^{iz}}{(z^2+1)(z^2+4)} \right]. \end{aligned}$$

Take the limit now as $R \rightarrow \infty$. The integral over C_R consequently tends to zero. Proof for this statement will be given at the end.

$$\int_{-\infty}^{\infty} \frac{re^{ir}}{(r^2+1)(r^2+4)} dr = 2\pi i \left[\operatorname{Res}_{z=i} \frac{ze^{iz}}{(z^2+1)(z^2+4)} + \operatorname{Res}_{z=2i} \frac{ze^{iz}}{(z^2+1)(z^2+4)} \right]$$

The denominator can be written as $(z^2+1)(z^2+4) = (z+i)(z-i)(z+2i)(z-2i)$. From this we see that the multiplicities of the $z-i$ and $z-2i$ factors are both 1. The residues at $z=i$ and $z=2i$ can then be calculated by

$$\begin{aligned} \operatorname{Res}_{z=i} \frac{ze^{iz}}{(z^2+1)(z^2+4)} &= \phi_1(i) \\ \operatorname{Res}_{z=2i} \frac{ze^{iz}}{(z^2+1)(z^2+4)} &= \phi_2(2i), \end{aligned}$$

where $\phi_1(z)$ and $\phi_2(z)$ are equal to $f(z)$ without the $z-i$ and $z-2i$ factors, respectively.

$$\begin{aligned} \phi_1(z) &= \frac{ze^{iz}}{(z+i)(z+2i)(z-2i)} \Rightarrow \phi_1(i) = \frac{1}{6}e^{-1} \\ \phi_2(z) &= \frac{ze^{iz}}{(z+i)(z-i)(z+2i)} \Rightarrow \phi_2(2i) = -\frac{1}{6}e^{-2} \end{aligned}$$

So then

$$\begin{aligned} \operatorname{Res}_{z=i} \frac{ze^{iz}}{(z^2+1)(z^2+4)} &= \frac{1}{6e} \\ \operatorname{Res}_{z=2i} \frac{ze^{iz}}{(z^2+1)(z^2+4)} &= -\frac{1}{6e^2} \end{aligned}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{re^{ir}}{(r^2+1)(r^2+4)} dr &= 2\pi i \left(\frac{1}{6e} - \frac{1}{6e^2} \right) \\ \int_{-\infty}^{\infty} \frac{r \cos r + ir \sin r}{(r^2+1)(r^2+4)} dr &= \frac{\pi i}{3} \left(\frac{1}{e} - \frac{1}{e^2} \right) \\ \int_{-\infty}^{\infty} \frac{r \cos r}{(r^2+1)(r^2+4)} dr + i \int_{-\infty}^{\infty} \frac{r \sin r}{(r^2+1)(r^2+4)} dr &= \frac{i\pi}{3} \left(\frac{e-1}{e^2} \right). \end{aligned}$$

Match the real and imaginary parts of both sides.

$$\int_{-\infty}^{\infty} \frac{r \cos r}{(r^2 + 1)(r^2 + 4)} dr = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{r \sin r}{(r^2 + 1)(r^2 + 4)} dr = \frac{\pi}{3} \left(\frac{e - 1}{e^2} \right)$$

Therefore, changing the dummy integration variable to x ,

$$\boxed{\int_{-\infty}^{\infty} \frac{x \sin x dx}{(x^2 + 1)(x^2 + 4)} = \frac{\pi}{3} \left(\frac{e - 1}{e^2} \right).}$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \rightarrow \infty$. The parameterization of the semicircular arc in Fig. 99 is $z = Re^{i\theta}$, where θ goes from 0 to π .

$$\begin{aligned} \int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)} dz &= \int_0^\pi \frac{Re^{i\theta} e^{iRe^{i\theta}}}{[(Re^{i\theta})^2+1][(Re^{i\theta})^2+4]} (Rie^{i\theta} d\theta) \\ &= \int_0^\pi \frac{e^{iR(\cos\theta+i\sin\theta)}}{(R^2e^{i2\theta}+1)(R^2e^{i2\theta}+4)} (R^2ie^{i2\theta} d\theta) \\ &= \int_0^\pi \frac{e^{iR\cos\theta} e^{-R\sin\theta}}{(R^2e^{i2\theta}+1)(R^2e^{i2\theta}+4)} (R^2ie^{i2\theta} d\theta) \end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned} \left| \int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)} dz \right| &= \left| \int_0^\pi \frac{e^{iR\cos\theta} e^{-R\sin\theta}}{(R^2e^{i2\theta}+1)(R^2e^{i2\theta}+4)} (R^2ie^{i2\theta} d\theta) \right| \\ &\leq \int_0^\pi \left| \frac{e^{iR\cos\theta} e^{-R\sin\theta}}{(R^2e^{i2\theta}+1)(R^2e^{i2\theta}+4)} (R^2ie^{i2\theta}) \right| d\theta \\ &= \int_0^\pi \frac{|e^{iR\cos\theta}| |e^{-R\sin\theta}|}{|R^2e^{i2\theta}+1| |R^2e^{i2\theta}+4|} |R^2ie^{i2\theta}| d\theta \\ &= \int_0^\pi \frac{e^{-R\sin\theta}}{|R^2e^{i2\theta}+1| |R^2e^{i2\theta}+4|} R^2 d\theta \\ &\leq \int_0^\pi \frac{e^{-R\sin\theta}}{(|R^2e^{i2\theta}|-|1|)(|R^2e^{i2\theta}|-|4|)} R^2 d\theta \\ &= \int_0^\pi \frac{e^{-R\sin\theta}}{(R^2-1)(R^2-4)} R^2 d\theta \\ &= \int_0^\pi \frac{e^{-R\sin\theta}}{\left(1-\frac{1}{R^2}\right)\left(1-\frac{4}{R^2}\right)} \frac{d\theta}{R^2} \end{aligned}$$

Now take the limit of both sides as $R \rightarrow \infty$.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)} dz \right| \leq \lim_{R \rightarrow \infty} \int_0^\pi \frac{e^{-R\sin\theta}}{\left(1-\frac{1}{R^2}\right)\left(1-\frac{4}{R^2}\right)} \frac{d\theta}{R^2}$$

Because the limits of integration do not depend on R , the limit may be brought inside the integral.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)} dz \right| \leq \int_0^\pi \lim_{R \rightarrow \infty} \frac{e^{-R\sin\theta}}{\left(1-\frac{1}{R^2}\right)\left(1-\frac{4}{R^2}\right)} \frac{d\theta}{R^2}$$

Since θ lies between 0 and π , the sine of θ is positive. Thus, the exponent of e tends to $-\infty$, and the integral tends to zero.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)} dz \right| \leq 0$$

The magnitude of a number cannot be negative, and the only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)} dz \right| = 0 \quad \rightarrow \quad \lim_{R \rightarrow \infty} \int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)} dz = 0.$$