Exercise 7

Use residues to evaluate the integrals in Exercises 6 and 7.

$$\int_0^\infty \frac{x^3 \sin x \, dx}{(x^2+1)(x^2+9)}.$$

Solution

The integrand is an even function of x, so the interval of integration can be extended to $(-\infty, \infty)$ as long as the integral is divided by 2.

$$\int_0^\infty \frac{x^3 \sin x \, dx}{(x^2+1)(x^2+9)} = \int_{-\infty}^\infty \frac{x^3 \sin x \, dx}{2(x^2+1)(x^2+9)}$$

In order to evaluate the integral, consider the corresponding function in the complex plane,

$$f(z) = \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)},$$

and the contour in Fig. 99. Singularities occur where the denominator is equal to zero.

$$\begin{gathered} 2(z^2+1)(z^2+9) = 0 \\ z^2+1=0 \quad \text{or} \quad z^2+9=0 \\ z = \pm i \quad \text{or} \quad z = \pm 3i \end{gathered}$$

The singular points of interest to us are the ones that lie within the closed contour, $z = i$ and $z = 3i$.

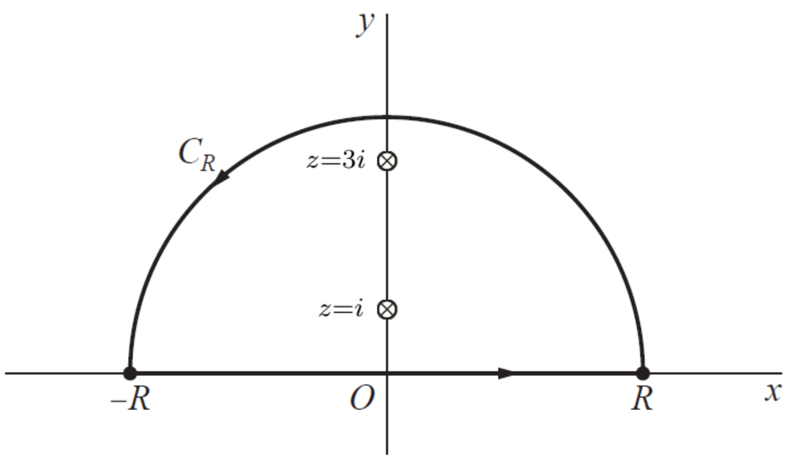

Figure 1: This is Fig. 99 with the singularities at $z = i$ and $z = 3i$ marked.

According to Cauchy's residue theorem, the integral of $z^3 e^{iz}/[2(z^2+1)(z^2+9)]$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)} \, dz = 2\pi i \left[\operatorname*{Res}_{z=i} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)} + \operatorname*{Res}_{z=3i} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)} \right]$$

This closed loop integral is the sum of two integrals, one over each arc in the loop.

$$\int_L \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)}\,dz + \int_{C_R} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)}\,dz = 2\pi i\left[\operatorname*{Res}_{z=i} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)} + \operatorname*{Res}_{z=3i} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)}\right]$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L:&\quad z = r, && r=-R \quad\to\quad r=R \\ C_R:&\quad z = Re^{i\theta}, && \theta=0 \quad\to\quad \theta=\pi \end{aligned}$$

As a result,

$$\int_{-R}^{R} \frac{r^3 e^{ir}}{2(r^2+1)(r^2+9)}\,dr + \int_{C_R} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)}\,dz = 2\pi i\left[\operatorname*{Res}_{z=i} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)} + \operatorname*{Res}_{z=3i} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)}\right].$$

Take the limit now as $R \to \infty$. The integral over C_R consequently tends to zero. Proof for this statement will be given at the end.

$$\int_{-\infty}^{\infty} \frac{r^3 e^{ir}}{2(r^2+1)(r^2+9)}\,dr = 2\pi i\left[\operatorname*{Res}_{z=i} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)} + \operatorname*{Res}_{z=3i} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)}\right]$$

The denominator can be written as $2(z^2+1)(z^2+9) = 2(z+i)(z-i)(z+3i)(z-3i)$. From this we see that the multiplicities of the $z-i$ and $z-3i$ factors are both 1. The residues at $z=i$ and $z=3i$ can then be calculated by

$$\begin{aligned} \operatorname*{Res}_{z=i} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)} &= \phi_1(i) \\ \operatorname*{Res}_{z=3i} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)} &= \phi_2(3i), \end{aligned}$$

where $\phi_1(z)$ and $\phi_2(z)$ are equal to $f(z)$ without the $z-i$ and $z-3i$ factors, respectively.

$$\begin{aligned} \phi_1(z) &= \frac{z^3 e^{iz}}{2(z+i)(z+3i)(z-3i)} \quad\Rightarrow\quad \phi_1(i) = -\frac{1}{32}e^{-1} \\ \phi_2(z) &= \frac{z^3 e^{iz}}{2(z+i)(z-i)(z+3i)} \quad\Rightarrow\quad \phi_2(3i) = \frac{9}{32}e^{-3} \end{aligned}$$

So then

$$\begin{aligned} \operatorname*{Res}_{z=i} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)} &= -\frac{1}{32}e^{-1} \\ \operatorname*{Res}_{z=3i} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)} &= \frac{9}{32}e^{-3} \end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{r^3 e^{ir}}{2(r^2+1)(r^2+9)}\,dr &= 2\pi i\left(-\frac{1}{32}e^{-1}+\frac{9}{32}e^{-3}\right)\\ \int_{-\infty}^{\infty} \frac{r^3\cos r + ir^3\sin r}{2(r^2+1)(r^2+9)}\,dr &= \frac{\pi i}{16}\left(\frac{9}{e^3}-\frac{1}{e}\right)\\ \int_{-\infty}^{\infty} \frac{r^3\cos r}{2(r^2+1)(r^2+9)}\,dr + i\int_{-\infty}^{\infty} \frac{r^3\sin r}{2(r^2+1)(r^2+9)}\,dr &= \frac{i\pi}{16}\left(\frac{9-e^2}{e^3}\right).\end{aligned}$$

Match the real and imaginary parts of both sides.

$$\int_{-\infty}^{\infty} \frac{r^3\cos r}{2(r^2+1)(r^2+9)}\,dr = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{r^3\sin r}{2(r^2+1)(r^2+9)}\,dr = \frac{\pi}{16}\left(\frac{9-e^2}{e^3}\right)$$

Therefore, changing the dummy integration variable to x,

$$\boxed{\int_0^{\infty} \frac{x^3\sin x\,dx}{(x^2+1)(x^2+9)} = \frac{\pi}{16}\left(\frac{9-e^2}{e^3}\right).}$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \to \infty$. The parameterization of the semicircular arc in Fig. 99 is $z = Re^{i\theta}$, where θ goes from 0 to π.

$$\begin{aligned}\int_{C_R} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)}\,dz &= \int_0^\pi \frac{(Re^{i\theta})^3 e^{iRe^{i\theta}}}{2[(Re^{i\theta})^2+1][(Re^{i\theta})^2+9]}(Rie^{i\theta}\,d\theta) \\ &= \int_0^\pi \frac{e^{iR(\cos\theta+i\sin\theta)}}{2(R^2e^{i2\theta}+1)(R^2e^{i2\theta}+9)}(R^4ie^{i4\theta}\,d\theta) \\ &= \int_0^\pi \frac{e^{iR\cos\theta}e^{-R\sin\theta}}{2(R^2e^{i2\theta}+1)(R^2e^{i2\theta}+9)}(R^4ie^{i4\theta}\,d\theta)\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}\left|\int_{C_R} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)}\,dz\right| &= \left|\int_0^\pi \frac{e^{iR\cos\theta}e^{-R\sin\theta}}{2(R^2e^{i2\theta}+1)(R^2e^{i2\theta}+9)}(R^4ie^{i4\theta}\,d\theta)\right| \\ &\le \int_0^\pi \left|\frac{e^{iR\cos\theta}e^{-R\sin\theta}}{2(R^2e^{i2\theta}+1)(R^2e^{i2\theta}+9)}(R^4ie^{i4\theta})\right| d\theta \\ &= \int_0^\pi \frac{\left|e^{iR\cos\theta}\right|\left|e^{-R\sin\theta}\right|}{2\left|R^2e^{i2\theta}+1\right|\left|R^2e^{i2\theta}+9\right|}\left|R^4ie^{i4\theta}\right| d\theta \\ &= \int_0^\pi \frac{e^{-R\sin\theta}}{2\left|R^2e^{i2\theta}+1\right|\left|R^2e^{i2\theta}+9\right|}R^4\,d\theta \\ &\le \int_0^\pi \frac{e^{-R\sin\theta}}{(|R^2e^{i2\theta}|-|1|)(|R^2e^{i2\theta}|-|9|)}\frac{R^4}{2}\,d\theta \\ &= \int_0^\pi \frac{e^{-R\sin\theta}}{(R^2-1)(R^2-9)}\frac{R^4}{2}\,d\theta \\ &= \int_0^\pi \frac{e^{-R\sin\theta}}{\left(1-\frac{1}{R^2}\right)\left(1-\frac{9}{R^2}\right)}\frac{d\theta}{2}\end{aligned}$$

Now take the limit of both sides as $R \to \infty$.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)}\,dz\right| \le \lim_{R\to\infty}\int_0^\pi \frac{e^{-R\sin\theta}}{\left(1-\frac{1}{R^2}\right)\left(1-\frac{9}{R^2}\right)}\frac{d\theta}{2}$$

Because the limits of integration do not depend on R, the limit may be brought inside the integral.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)}\,dz\right| \le \int_0^\pi \lim_{R\to\infty}\frac{e^{-R\sin\theta}}{\left(1-\frac{1}{R^2}\right)\left(1-\frac{9}{R^2}\right)}\frac{d\theta}{2}$$

Since θ lies between 0 and π, the sine of θ is positive. Thus, the exponent of e tends to $-\infty$, and the integral tends to zero.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)}\,dz\right| \le 0$$

The magnitude of a number cannot be negative, and the only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)}\,dz\right| = 0 \quad\to\quad \lim_{R\to\infty}\int_{C_R} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)}\,dz = 0.$$