Exercise 4

Derive the integration formula

$$\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}\,dx = \frac{2\pi}{\sqrt{3}} \cdot \frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b} \qquad (a > b > 0).$$

using the function

$$f(z) = \frac{z^{1/3}}{(z+a)(z+b)} = \frac{e^{(1/3)\log z}}{(z+a)(z+b)} \qquad (|z| > 0,\ 0 < \arg z < 2\pi)$$

and a closed contour similar to the one in Fig. 110 (Sec. 91), but where

$$\rho < b < a < R.$$

Solution

In order to evaluate this integral, consider the given function in the complex plane and the contour in Fig. 110. Singularities occur where the denominator is equal to zero.

$$(z+a)(z+b) = 0$$
$$z = -a \quad \text{or} \quad z = -b$$

Since $z^{1/3}$ can be written in terms of $\log z$, a branch cut for the function needs to be chosen.

$$z^{1/3} = \exp\left(\frac{1}{3}\log z\right)$$

It has been chosen here to be the axis of positive real numbers.

$$= \exp\left[\frac{1}{3}(\ln r + i\theta)\right], \quad (|z| > 0,\ 0 < \theta < 2\pi)$$
$$= r^{1/3}e^{i\theta/3},$$

where $r = |z|$ is the magnitude of z and $\theta = \arg z$ is the argument of z.

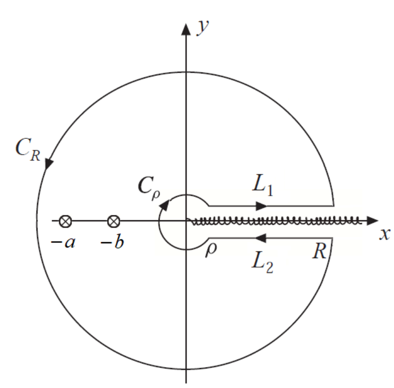

Figure 1: This is essentially Fig. 110 with the singularities at $z = -a$ and $z = -b$ marked. The squiggly line represents the branch cut $(|z| > 0,\ 0 < \theta < 2\pi)$.

According to Cauchy's residue theorem, the integral of $z^{1/3}/[(z+a)(z+b)]$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{z^{1/3}}{(z+a)(z+b)}\,dz = 2\pi i\left[\operatorname*{Res}_{z=-a}\frac{z^{1/3}}{(z+a)(z+b)} + \operatorname*{Res}_{z=-b}\frac{z^{1/3}}{(z+a)(z+b)}\right]$$

This closed loop integral is the sum of four integrals, one over each arc in the loop.

$$\int_{L_1}\frac{z^{1/3}}{(z+a)(z+b)}\,dz + \int_{L_2}\frac{z^{1/3}}{(z+a)(z+b)}\,dz + \int_{C_\rho}\frac{z^{1/3}}{(z+a)(z+b)}\,dz + \int_{C_R}\frac{z^{1/3}}{(z+a)(z+b)}\,dz$$
$$= 2\pi i\left[\operatorname*{Res}_{z=-a}\frac{z^{1/3}}{(z+a)(z+b)} + \operatorname*{Res}_{z=-b}\frac{z^{1/3}}{(z+a)(z+b)}\right] \quad (1)$$

The parameterizations for the arcs are as follows.

$$\begin{aligned}
L_1: &\quad z = re^{i0}, && r=\rho \quad\to\quad r=R\\
L_2: &\quad z = re^{i2\pi}, && r=R \quad\to\quad r=\rho\\
C_\rho: &\quad z = \rho e^{i\theta}, && \theta=2\pi \quad\to\quad \theta=0\\
C_R: &\quad z = Re^{i\theta}, && \theta=0 \quad\to\quad \theta=2\pi
\end{aligned}$$

As a result,

$$\begin{aligned}
\int_{L_1}\frac{z^{1/3}}{(z+a)(z+b)}\,dz + \int_{L_2}\frac{z^{1/3}}{(z+a)(z+b)}\,dz &= \int_\rho^R \frac{(re^{i0})^{1/3}}{(re^{i0}+a)(re^{i0}+b)}\,(dr\,e^{i0}) + \int_R^\rho \frac{(re^{i2\pi})^{1/3}}{(re^{i2\pi}+a)(re^{i2\pi}+b)}\,(dr\,e^{i2\pi})\\
&= \int_\rho^R \frac{r^{1/3}}{(r+a)(r+b)}\,dr + \int_R^\rho \frac{r^{1/3}e^{i2\pi/3}}{(r+a)(r+b)}\,dr\\
&= \int_\rho^R \frac{r^{1/3}}{(r+a)(r+b)}\,dr - \int_\rho^R \frac{r^{1/3}e^{i2\pi/3}}{(r+a)(r+b)}\,dr\\
&= (1-e^{2i\pi/3})\int_\rho^R \frac{r^{1/3}}{(r+a)(r+b)}\,dr.
\end{aligned}$$

Substitute this formula into equation (1).

$$(1-e^{2i\pi/3})\int_\rho^R \frac{r^{1/3}}{(r+a)(r+b)}\,dr + \int_{C_\rho}\frac{z^{1/3}}{(z+a)(z+b)}\,dz + \int_{C_R}\frac{z^{1/3}}{(z+a)(z+b)}\,dz$$
$$= 2\pi i\left[\operatorname*{Res}_{z=-a}\frac{z^{1/3}}{(z+a)(z+b)} + \operatorname*{Res}_{z=-b}\frac{z^{1/3}}{(z+a)(z+b)}\right]$$

Take the limit now as $\rho\to 0$ and $R\to\infty$. The integral over C_ρ tends to zero, and the integral over C_R tends to zero. Proof for these statements will be given at the end.

$$(1-e^{2i\pi/3})\int_0^\infty \frac{r^{1/3}}{(r+a)(r+b)}\,dr = 2\pi i\left[\operatorname*{Res}_{z=-a}\frac{z^{1/3}}{(z+a)(z+b)} + \operatorname*{Res}_{z=-b}\frac{z^{1/3}}{(z+a)(z+b)}\right]$$

The multiplicities of $z+a$ and $z+b$ in the denominator are both 1, so the residues at $z=-a$ and $z=-b$ can be calculated by

$$\operatorname*{Res}_{z=-a} \frac{z^{1/3}}{(z+a)(z+b)} = \phi_1(-a)$$
$$\operatorname*{Res}_{z=-b} \frac{z^{1/3}}{(z+a)(z+b)} = \phi_2(-b),$$

where $\phi_1(z)$ and $\phi_2(z)$ are the same function as $f(z)$ without the factors, $z+a$ and $z+b$, respectively.

$$\phi_1(z) = \frac{z^{1/3}}{z+b} \quad \Rightarrow \quad \phi_1(-a) = \frac{(-a)^{1/3}}{-a+b} = \frac{(ae^{i\pi})^{1/3}}{-a+b} = -\frac{a^{1/3}e^{i\pi/3}}{a-b}$$
$$\phi_2(z) = \frac{z^{1/3}}{z+a} \quad \Rightarrow \quad \phi_2(-b) = \frac{(-b)^{1/3}}{-b+a} = \frac{(be^{i\pi})^{1/3}}{-b+a} = \frac{b^{1/3}e^{i\pi/3}}{a-b}$$

So then

$$\operatorname*{Res}_{z=-a} \frac{z^{1/3}}{(z+a)(z+b)} = -\frac{a^{1/3}e^{i\pi/3}}{a-b}$$
$$\operatorname*{Res}_{z=-b} \frac{z^{1/3}}{(z+a)(z+b)} = \frac{b^{1/3}e^{i\pi/3}}{a-b}$$

and

$$\begin{aligned}(1-e^{2i\pi/3})\int_0^\infty \frac{r^{1/3}}{(r+a)(r+b)}\,dr &= 2\pi i\left[-\frac{a^{1/3}e^{i\pi/3}}{a-b} + \frac{b^{1/3}e^{i\pi/3}}{a-b}\right] \\ &= \frac{2\pi i}{a-b}e^{i\pi/3}(-a^{1/3}+b^{1/3}).\end{aligned}$$

Divide both sides by $1-e^{2i\pi/3}$.

$$\begin{aligned}\int_0^\infty \frac{r^{1/3}}{(r+a)(r+b)}\,dr &= \frac{2\pi i}{a-b}\cdot\frac{e^{i\pi/3}}{1-e^{2i\pi/3}}(-a^{1/3}+b^{1/3}) \\ &= \frac{2\pi i}{a-b}\cdot\frac{1}{e^{-i\pi/3}-e^{i\pi/3}}(-a^{1/3}+b^{1/3}) \\ &= \frac{2\pi i}{a-b}\cdot\frac{1}{[-2i\sin(\pi/3)]}(-a^{1/3}+b^{1/3}) \\ &= \frac{2\pi}{a-b}\cdot\frac{1}{\sqrt{3}}(a^{1/3}-b^{1/3})\end{aligned}$$

Therefore, changing the dummy integration variable to x,

$$\boxed{\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}\,dx = \frac{2\pi}{\sqrt{3}}\cdot\frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b}.}$$

The Integral Over C_ρ

Our aim here is to show that the integral over C_ρ tends to zero in the limit as $\rho \to 0$. The parameterization of the small circular arc in Figure 1 is $z = \rho e^{i\theta}$, where θ goes from 2π to 0.

$$\begin{aligned}\int_{C_\rho} \frac{z^{1/3}}{(z+a)(z+b)}\,dz &= \int_{2\pi}^{0} \frac{(\rho e^{i\theta})^{1/3}}{(\rho e^{i\theta}+a)(\rho e^{i\theta}+b)}(\rho i e^{i\theta}\,d\theta) \\ &= \int_{2\pi}^{0} \frac{\rho^{4/3}}{(\rho e^{i\theta}+a)(\rho e^{i\theta}+b)}(ie^{4i\theta/3}\,d\theta)\end{aligned}$$

Take the limit of both sides as $\rho \to 0$.

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{z^{1/3}}{(z+a)(z+b)}\,dz = \lim_{\rho\to 0}\int_{2\pi}^{0} \frac{\rho^{4/3}}{(\rho e^{i\theta}+a)(\rho e^{i\theta}+b)}(ie^{4i\theta/3}\,d\theta)$$

The limits of integration are constant, so the limit may be brought inside the integral.

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{z^{1/3}}{(z+a)(z+b)}\,dz = \int_{2\pi}^{0} \lim_{\rho\to 0}\frac{\rho^{4/3}}{(\rho e^{i\theta}+a)(\rho e^{i\theta}+b)}(ie^{4i\theta/3}\,d\theta)$$

Because of $\rho^{4/3}$ in the numerator, the limit is zero. Therefore,

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{z^{1/3}}{(z+a)(z+b)}\,dz = 0.$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \to \infty$. The parameterization of the large circular arc in Figure 1 is $z = Re^{i\theta}$, where θ goes from 0 to 2π.

$$\begin{aligned}\int_{C_R} \frac{z^{1/3}}{(z+a)(z+b)}\,dz &= \int_{0}^{2\pi} \frac{(Re^{i\theta})^{1/3}}{(Re^{i\theta}+a)(Re^{i\theta}+b)}(Rie^{i\theta}\,d\theta) \\ &= \int_{0}^{2\pi} \frac{R^{4/3}}{(Re^{i\theta}+a)(Re^{i\theta}+b)}(ie^{4i\theta/3}\,d\theta) \\ &= \int_{0}^{2\pi} \frac{R^{4/3}}{R^2\left(e^{i\theta}+\frac{a}{R}\right)\left(e^{i\theta}+\frac{b}{R}\right)}(ie^{4i\theta/3}\,d\theta) \\ &= \int_{0}^{2\pi} \frac{1}{R^{2/3}}\frac{1}{\left(e^{i\theta}+\frac{a}{R}\right)\left(e^{i\theta}+\frac{b}{R}\right)}(ie^{4i\theta/3}\,d\theta)\end{aligned}$$

Take the limit of both sides as $R \to \infty$. Since the limits of integration are constant, the limit may be brought inside the integral.

$$\lim_{R\to\infty}\int_{C_R} \frac{z^{1/3}}{(z+a)(z+b)}\,dz = \int_{0}^{2\pi} \lim_{R\to\infty}\frac{1}{R^{2/3}}\frac{1}{\left(e^{i\theta}+\frac{a}{R}\right)\left(e^{i\theta}+\frac{b}{R}\right)}(ie^{4i\theta/3}\,d\theta)$$

Because of $R^{2/3}$ in the denominator, the limit is zero. Therefore,

$$\lim_{R\to\infty}\int_{C_R} \frac{z^{1/3}}{(z+a)(z+b)}\,dz = 0.$$