# Exercise 5

The ***beta function*** is this function of two real variables:

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt \qquad (p>0,\ q>0).$$

Make the substitution $t = 1/(x+1)$ and use the result obtained in the example in Sec. 91 to show that

$$B(p,1-p) = \frac{\pi}{\sin(p\pi)} \qquad (0<p<1).$$

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**Solution**

Making the prescribed substitution, we have

$$t = \frac{1}{x+1} \quad \rightarrow \quad \begin{cases} x = \dfrac{1}{t} - 1 \\ 1-t = \dfrac{x}{x+1} \end{cases}$$
$$dt = -\frac{1}{(x+1)^2}\,dx,$$

so the beta function becomes

$$\begin{aligned} B(p,q) &= \int_\infty^0 \left(\frac{1}{x+1}\right)^{p-1} \left(\frac{x}{x+1}\right)^{q-1} \left[-\frac{1}{(x+1)^2}\,dx\right] \\ &= \int_0^\infty \frac{x^{q-1}}{(x+1)^{p+q-2}} \left[\frac{1}{(x+1)^2}\,dx\right] \\ &= \int_0^\infty \frac{x^{q-1}}{(x+1)^{p+q}}\,dx. \end{aligned}$$

Now substitute $q = 1-p$.

$$\begin{aligned} B(p,1-p) &= \int_0^\infty \frac{x^{(1-p)-1}}{(x+1)^{p+(1-p)}}\,dx \\ &= \int_0^\infty \frac{x^{-p}}{x+1}\,dx \end{aligned}$$

In order to evaluate this integral, consider the corresponding function in the complex plane,

$$f(z) = \frac{z^{-p}}{z+1},$$

and the contour in Figure 1. Singularities occur where the denominator is equal to zero.

$$\begin{aligned} z+1 &= 0 \\ z &= -1 \end{aligned}$$

Because $z^{-p}$ can be written in terms of the logarithm function, a branch cut has to be chosen.

$$z^{-p} = \exp(-p\log z)$$

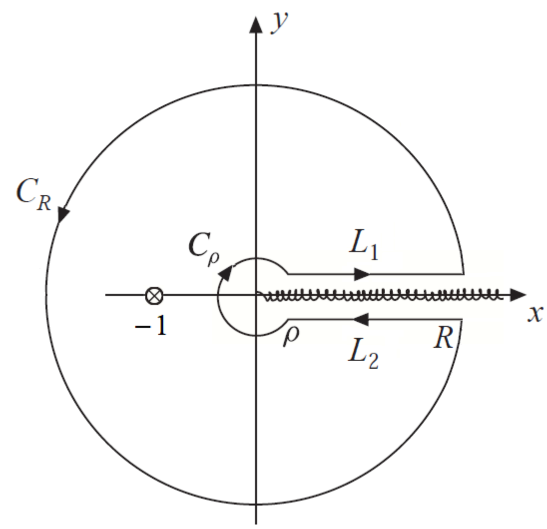


Figure 1: This is essentially Fig. 110 in the textbook with the singularity at $z = -1$ marked. The squiggly line represents the branch cut $(|z| > 0,\ 0 < \theta < 2\pi)$.

We choose it to be the axis of positive real numbers so that the contour doesn't have to be indented more than once.

$$\begin{aligned} z^{-p} &= \exp\left[-p\left(\ln r + i\theta\right)\right], \quad (|z| > 0,\ 0 < \theta < 2\pi) \\ &= r^{-p} e^{-ip\theta}, \end{aligned}$$

where $r = |z|$ is the magnitude of $z$ and $\theta = \arg z$ is the argument of $z$. According to Cauchy's residue theorem, the integral of $z^{-p}/(z+1)$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{z^{-p}}{z+1}\, dz = 2\pi i \operatorname*{Res}_{z=-1} \frac{z^{-p}}{z+1}$$

This closed loop integral is the sum of four integrals, one over each arc in the loop.

$$\int_{L_1} \frac{z^{-p}}{z+1}\, dz + \int_{L_2} \frac{z^{-p}}{z+1}\, dz + \int_{C_\rho} \frac{z^{-p}}{z+1}\, dz + \int_{C_R} \frac{z^{-p}}{z+1}\, dz = 2\pi i \operatorname*{Res}_{z=-1} \frac{z^{-p}}{z+1}$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L_1 &: \quad z = re^{i0}, && r = \rho \quad \to \quad r = R \\ L_2 &: \quad z = re^{i2\pi}, && r = R \quad \to \quad r = \rho \\ C_\rho &: \quad z = \rho e^{i\theta}, && \theta = 2\pi \quad \to \quad \theta = 0 \\ C_R &: \quad z = Re^{i\theta}, && \theta = 0 \quad \to \quad \theta = 2\pi \end{aligned}$$

As a result,

$$\begin{aligned}
2\pi i \operatorname*{Res}_{z=-1} \frac{z^{-p}}{z+1} &= \int_\rho^R \frac{(re^{i0})^{-p}}{re^{i0}+1}(dr\, e^{i0}) + \int_R^\rho \frac{(re^{i2\pi})^{-p}}{re^{i2\pi}+1}(dr\, e^{i2\pi}) + \int_{C_\rho} \frac{z^{-p}}{z+1}\, dz + \int_{C_R} \frac{z^{-p}}{z+1}\, dz \\
&= \int_\rho^R \frac{r^{-p}}{r+1}\, dr + \int_R^\rho \frac{r^{-p}e^{-i2p\pi}}{r+1}\, dr + \int_{C_\rho} \frac{z^{-p}}{z+1}\, dz + \int_{C_R} \frac{z^{-p}}{z+1}\, dz \\
&= \int_\rho^R \frac{r^{-p}}{r+1}\, dr - \int_\rho^R \frac{r^{-p}e^{-i2p\pi}}{r+1}\, dr + \int_{C_\rho} \frac{z^{-p}}{z+1}\, dz + \int_{C_R} \frac{z^{-p}}{z+1}\, dz \\
&= (1-e^{-i2p\pi}) \int_\rho^R \frac{r^{-p}}{r+1}\, dr + \int_{C_\rho} \frac{z^{-p}}{z+1}\, dz + \int_{C_R} \frac{z^{-p}}{z+1}\, dz.
\end{aligned}$$

Take the limit now as $\rho \to 0$ and $R \to \infty$. As long as $0 < p < 1$ the integral over $C_\rho$ tends to zero, and as long as $p > 0$ the integral over $C_R$ tends to zero. Proof for these statements will be given at the end.

$$(1-e^{-i2p\pi}) \int_0^\infty \frac{r^{-p}}{r+1}\, dr = 2\pi i \operatorname*{Res}_{z=-1} \frac{z^{-p}}{z+1}$$

The residue at $z = -1$ can be calculated by evaluating the numerator at $-1$.

$$\operatorname*{Res}_{z=-1} \frac{z^{-p}}{z+1} = (-1)^{-p} = (e^{i\pi})^{-p} = e^{-ip\pi}$$

So then

$$(1-e^{-i2p\pi}) \int_0^\infty \frac{r^{-p}}{r+1}\, dr = 2\pi i e^{-ip\pi}.$$

Divide both sides by $1 - e^{-i2p\pi}$ and simplify.

$$\begin{aligned}
\int_0^\infty \frac{r^{-p}}{r+1}\, dr &= 2\pi i \frac{e^{-ip\pi}}{1-e^{-i2p\pi}} \\
&= 2\pi i \frac{1}{e^{ip\pi} - e^{-ip\pi}} \\
&= 2\pi i \frac{1}{2i \sin p\pi} \\
&= \frac{\pi}{\sin p\pi}
\end{aligned}$$

Changing the dummy integration variable to $x$,

$$\int_0^\infty \frac{x^{-p}}{x+1}\, dx = \frac{\pi}{\sin p\pi}, \quad 0 < p < 1.$$

Therefore,

$$\boxed{B(p, 1-p) = \frac{\pi}{\sin p\pi}, \quad 0 < p < 1.}$$

**The Integral Over $C_\rho$**

Our aim here is to show that the integral over $C_\rho$ tends to zero in the limit as $\rho \to 0$. The parameterization of the small circular arc in Figure 1 is $z = \rho e^{i\theta}$, where $\theta$ goes from $2\pi$ to 0.

$$\begin{aligned}\int_{C_\rho} \frac{z^{-p}}{z+1}\,dz &= \int_{2\pi}^{0} \frac{(\rho e^{i\theta})^{-p}}{\rho e^{i\theta}+1}(\rho i e^{i\theta}\,d\theta) \\ &= \int_{2\pi}^{0} \frac{\rho^{1-p}}{\rho e^{i\theta}+1}[ie^{i\theta(1-p)}\,d\theta]\end{aligned}$$

Take the limit of both sides as $\rho \to 0$.

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{z^{-p}}{z+1}\,dz = \lim_{\rho\to 0}\int_{2\pi}^{0} \frac{\rho^{1-p}}{\rho e^{i\theta}+1}[ie^{i\theta(1-p)}\,d\theta]$$

The limits of integration are constant, so the limit may be brought inside the integral.

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{z^{-p}}{z+1}\,dz = \int_{2\pi}^{0} \lim_{\rho\to 0}\frac{\rho^{1-p}}{\rho e^{i\theta}+1}[ie^{i\theta(1-p)}\,d\theta]$$

Provided that $0 < p < 1$, $\rho^{1-p}$ tends to zero. Therefore,

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{z^{-p}}{z+1}\,dz = 0.$$

**The Integral Over $C_R$**

Our aim here is to show that the integral over $C_R$ tends to zero in the limit as $R \to \infty$. The parameterization of the large circular arc in Figure 1 is $z = Re^{i\theta}$, where $\theta$ goes from 0 to $2\pi$.

$$\begin{aligned}\int_{C_R} \frac{z^{-p}}{z+1}\,dz &= \int_{0}^{2\pi} \frac{(Re^{i\theta})^{-p}}{Re^{i\theta}+1}(Rie^{i\theta}\,d\theta) \\ &= \int_{0}^{2\pi} \frac{R^{1-p}}{Re^{i\theta}+1}[ie^{i\theta(1-p)}\,d\theta] \\ &= \int_{0}^{2\pi} \frac{R^{-p}}{e^{i\theta}+\frac{1}{R}}[ie^{i\theta(1-p)}\,d\theta]\end{aligned}$$

Take the limit of both sides as $R \to \infty$. Since the limits of integration are constant, the limit may be brought inside the integral.

$$\lim_{R\to\infty}\int_{C_R} \frac{z^{-p}}{z+1}\,dz = \int_{0}^{2\pi} \lim_{R\to\infty}\frac{R^{-p}}{e^{i\theta}+\frac{1}{R}}[ie^{i\theta(1-p)}\,d\theta]$$

Provided that $p > 0$, the limit is zero because of $R^{-p}$ in the numerator. Therefore,

$$\lim_{R\to\infty}\int_{C_R} \frac{z^{-p}}{z+1}\,dz = 0.$$