Exercise 2

Use residues to establish the following integration formula:

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \sqrt{2}\pi.$$

Solution

Before we get started with solving this integral, we want the limits of integration to be from 0 to 2π, so let $x = \theta + \pi$. Then $dx = d\theta$ and

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} &= \int_0^{2\pi} \frac{dx}{1+\sin^2(x-\pi)} \\
&= \int_0^{2\pi} \frac{dx}{1+(-1)^2\sin^2 x} \\
&= \int_0^{2\pi} \frac{dx}{1+\sin^2 x}.
\end{aligned}$$

Because the integral now goes from 0 to 2π and the integrand is in terms of $\sin x$, we can make the substitution, $z = e^{ix}$. Euler's formula states that $e^{ix} = \cos x + i\sin x$, so we can write $\sin x$ and dx in terms of z and dz, respectively.

$$\sin x = \frac{z - z^{-1}}{2i} \quad \text{and} \quad dx = \frac{dz}{iz}.$$

The integral becomes

$$\begin{aligned}
\int_0^{2\pi} \frac{dx}{1+\sin^2 x} &= \int_C \frac{1}{1+\left(\frac{z-z^{-1}}{2i}\right)^2} \frac{dz}{iz} \\
&= \int_C \frac{1}{\frac{3}{2} - \frac{1}{4z^2} - \frac{z^2}{4}} \frac{-4iz\,dz}{4z^2} \\
&= \int_C \frac{4iz\,dz}{z^4 - 6z^2 + 1} \\
&= \int_C \frac{4iz\,dz}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} \\
&= \int_C f(z)\,dz,
\end{aligned}$$

where the contour C is the positively oriented unit circle centered at the origin and z_1, z_2, z_3, and z_4 are the zeros of $z^4 - 6z^2 + 1$.

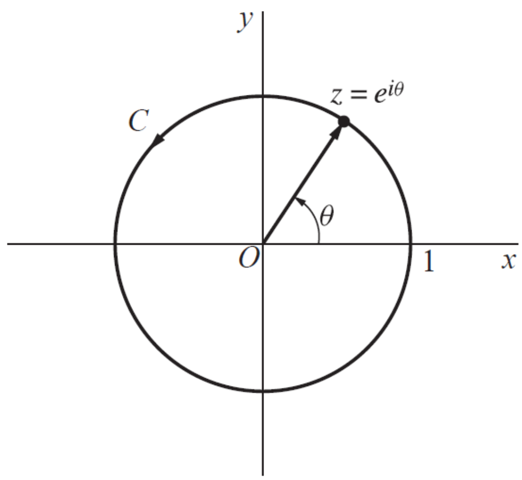

Figure 1: This figure illustrates the unit circle in the complex plane, where $z = x + iy$.

According to Cauchy's residue theorem, this contour integral is $2\pi i$ times the sum of the residues of $f(z)$ at the singular points inside the contour.

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

$f(z)$ has four singular points, $z_1 = 1+\sqrt{2} \approx 2.414$, $z_2 = 1-\sqrt{2} \approx -0.414$, $z_3 = -1+\sqrt{2} \approx 0.414$, and $z_4 = -1-\sqrt{2} \approx -2.414$. Since z_1 and z_4 lie outside the unit circle, they make no contribution to the integral. However, z_2 and z_3 do lie inside the circle, so we have to evaluate the residues of $f(z)$ at these points. Because z_2 and z_3 are simple poles, the residues can be written as

$$\operatorname*{Res}_{z=z_2} f(z) = \phi_1(z_2)$$
$$\operatorname*{Res}_{z=z_3} f(z) = \phi_2(z_3),$$

where $\phi_1(z)$ and $\phi_2(z)$ are determined from $f(z)$.

$$f(z) = \frac{\phi_1(z)}{z-z_2} \quad \to \quad \phi_1(z) = \frac{4iz}{(z-z_1)(z-z_3)(z-z_4)}$$
$$f(z) = \frac{\phi_2(z)}{z-z_3} \quad \to \quad \phi_2(z) = \frac{4iz}{(z-z_1)(z-z_2)(z-z_4)}$$

So

$$\operatorname*{Res}_{z=z_2} f(z) = \phi_1(z_2) = \frac{4iz_2}{(z_2-z_1)(z_2-z_3)(z_2-z_4)} = \frac{1}{2i\sqrt{2}}$$
$$\operatorname*{Res}_{z=z_3} f(z) = \phi_2(z_3) = \frac{4iz_3}{(z_3-z_1)(z_3-z_2)(z_3-z_4)} = \frac{1}{2i\sqrt{2}}.$$

This means that

$$\int_C f(z)\,dz = 2\pi i\left(\frac{1}{2i\sqrt{2}} + \frac{1}{2i\sqrt{2}}\right) = \sqrt{2}\pi.$$

Therefore,

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \sqrt{2}\pi.$$