Exercise 3

Use residues to establish the following integration formula:

$$\int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{5 - 4\cos 2\theta} = \frac{3\pi}{8}.$$

Solution

Because the integral goes from 0 to 2π and the integrand is in terms of cosine, we can make the substitution, $z = e^{i\theta}$. Using de Moivre's formula, $(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$, we can write $\cos^2 3\theta$ and $\cos 2\theta$ in terms of z and $d\theta$ in terms dz.

$$\cos 3\theta = \frac{z^3 + z^{-3}}{2} \quad \text{and} \quad \cos 2\theta = \frac{z^2 + z^{-2}}{2} \quad \text{and} \quad d\theta = \frac{dz}{iz}.$$

The integral becomes

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{5 - 4\cos 2\theta} &= \int_C \frac{\left(\frac{z^3+z^{-3}}{2}\right)^2}{5 - 4\left(\frac{z^2+z^{-2}}{2}\right)} \frac{dz}{iz} \\
&= \int_C \frac{z^6 + 2 + z^{-6}}{5 - 2z^2 - 2z^{-2}} \frac{dz}{4iz} \\
&= \int_C \frac{z^{12} + 2z^6 + 1}{5 - 2z^2 - 2z^{-2}} \frac{dz}{4iz^7} \\
&= \int_C \frac{z^{12} + 2z^6 + 1}{2z^4 - 5z^2 + 2} \frac{i\,dz}{4z^5} \\
&= \int_C \frac{i(z^6+1)^2}{4z^5(2z^4 - 5z^2 + 2)} \, dz \\
&= \int_C \frac{i(z^6+1)^2}{8z^5(z-z_1)(z-z_2)(z-z_3)(z-z_4)} \, dz \\
&= \int_C f(z)\, dz,
\end{aligned}$$

where the contour C is the positively oriented unit circle centered at the origin and z_1, z_2, z_3, and z_4 are the zeros of $z^4 - \frac{5}{2}z^2 + 1$.

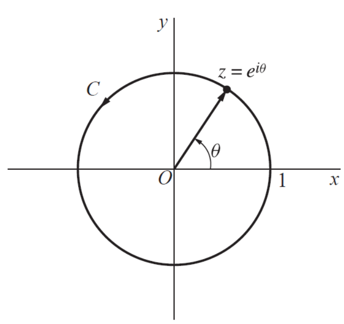

Figure 1: This figure illustrates the unit circle in the complex plane, where $z = x + iy$.

According to Cauchy's residue theorem, this contour integral is $2\pi i$ times the sum of the residues of $f(z)$ at the singular points inside the contour. That is,

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

$f(z)$ has five singular points, $z_1 = -\frac{1}{\sqrt{2}} \approx -0.707$, $z_2 = \frac{1}{\sqrt{2}} \approx 0.707$, $z_3 = -\sqrt{2} \approx -1.414$, $z_4 = \sqrt{2} \approx 1.414$, and 0. Since z_3 and z_4 lie outside the unit circle, they make no contribution to the integral. However, z_1, z_2, and 0 do lie inside the circle, so we have to evaluate the residues of $f(z)$ at these points. Because z_1 and z_2 are simple poles and 0 is a pole of order 5, the residues can be written as

$$\begin{aligned}
\operatorname*{Res}_{z=z_1} f(z) &= \phi_1(z_1) \\
\operatorname*{Res}_{z=z_2} f(z) &= \phi_2(z_2) \\
\operatorname*{Res}_{z=0} f(z) &= \frac{\phi^{(5-1)}(0)}{(5-1)!} = \frac{\phi''''(0)}{24}
\end{aligned}$$

where $\phi_1(z)$, $\phi_2(z)$, and $\phi(z)$ are determined from $f(z)$.

$$\begin{aligned}
f(z) &= \frac{\phi_1(z)}{z-z_1} \quad \rightarrow \quad \phi_1(z) = \frac{i(z^6+1)^2}{8z^5(z-z_2)(z-z_3)(z-z_4)} \\
f(z) &= \frac{\phi_2(z)}{z-z_2} \quad \rightarrow \quad \phi_2(z) = \frac{i(z^6+1)^2}{8z^5(z-z_1)(z-z_3)(z-z_4)} \\
f(z) &= \frac{\phi(z)}{z^5} \quad \rightarrow \quad \phi(z) = \frac{i(z^6+1)^2}{4(2z^4-5z^2+2)}
\end{aligned}$$

So

$$\begin{aligned}
\operatorname*{Res}_{z=z_1} f(z) &= \phi_1(z_1) = \frac{i(z_1^6+1)^2}{8z_1^5(z_1-z_2)(z_1-z_3)(z_1-z_4)} = -\frac{27}{64}i \\
\operatorname*{Res}_{z=z_2} f(z) &= \phi_2(z_2) = \frac{i(z_2^6+1)^2}{8z_2^5(z_2-z_1)(z_2-z_3)(z_2-z_4)} = -\frac{27}{64}i \\
\operatorname*{Res}_{z=0} f(z) &= \frac{\phi''''(0)}{24} = \frac{21}{32}i.
\end{aligned}$$

This means that

$$\int_C f(z)\,dz = 2\pi i\left(-\frac{27}{64}i - \frac{27}{64}i + \frac{21}{32}i\right) = 2\pi i\left(-\frac{3i}{16}\right) = \frac{3\pi}{8}.$$

Therefore,

$$\int_0^{2\pi} \frac{\cos^2 3\theta\,d\theta}{5-4\cos 2\theta} = \frac{3\pi}{8}.$$