Exercise 5

Use residues to establish the following integration formula:

$$\int_0^\pi \frac{d\theta}{(a+\cos\theta)^2} = \frac{a\pi}{(\sqrt{a^2-1})^3} \quad (a>1).$$

Solution

Before we get started with solving this integral, we want the limits of integration to be from 0 to 2π. Note that the integrand is even with respect to θ, so the integration interval can be extended to $[-\pi,\pi]$ so long as the integral is divided by 2. Then make the change of variables, $x=\theta+\pi$ and $dx=d\theta$ to achieve the desired limits.

$$\begin{aligned}\int_0^\pi \frac{d\theta}{(a+\cos\theta)^2} &= \frac{1}{2}\int_{-\pi}^{\pi} \frac{d\theta}{(a+\cos\theta)^2} \\ &= \frac{1}{2}\int_0^{2\pi} \frac{dx}{[a+\cos(x-\pi)]^2} \\ &= \frac{1}{2}\int_0^{2\pi} \frac{dx}{(a-\cos x)^2}\end{aligned}$$

Because the integral now goes from 0 to 2π and the integrand is in terms of $\cos x$, we can make the substitution, $z=e^{ix}$. Euler's formula states that $e^{ix}=\cos x+i\sin x$, so we can write $\cos x$ and dx in terms of z and dz, respectively.

$$\cos x = \frac{z+z^{-1}}{2} \quad \text{and} \quad dx = \frac{dz}{iz}.$$

The integral becomes

$$\begin{aligned}\frac{1}{2}\int_0^{2\pi} \frac{dx}{(a-\cos x)^2} &= \int_C \frac{1}{2}\frac{1}{\left[a-\left(\frac{z+z^{-1}}{2}\right)\right]^2}\frac{dz}{iz} \\ &= \int_C \frac{1}{2}\frac{4z^2}{(z^2-2az+1)^2}\frac{dz}{iz} \\ &= \int_C \frac{-2iz}{(z^2-2az+1)^2}\,dz \\ &= \int_C \frac{-2iz}{(z-z_1)^2(z-z_2)^2}\,dz \\ &= \int_C f(z)\,dz,\end{aligned}$$

where the contour C is the positively oriented unit circle centered at the origin and z_1 and z_2 are the zeros of $z^2-2az+1$.

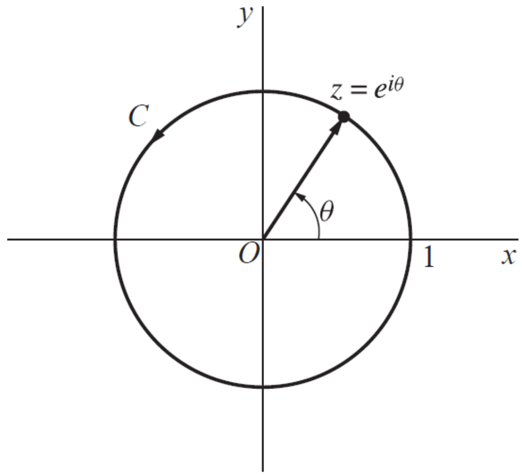

Figure 1: This figure illustrates the unit circle in the complex plane, where $z = x + iy$.

According to Cauchy's residue theorem, this contour integral is $2\pi i$ times the sum of the residues of $f(z)$ at the singular points inside the contour. That is,

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

$f(z)$ has two singular points,

$$z_1 = a - \sqrt{a^2-1}$$
$$z_2 = a + \sqrt{a^2-1}.$$

Since $a > 1$, z_2 lies outside the unit circle and thus makes no contribution to the integral. However, z_1 does lie inside the circle, so we have to evaluate the residue of $f(z)$ at this point. Because z_1 is a pole of order 2, the residue can be written as

$$\operatorname*{Res}_{z=z_1} f(z) = \frac{\phi^{(2-1)}(z_1)}{(2-1)!} = \phi'(z_1),$$

where $\phi(z)$ is determined from $f(z)$.

$$f(z) = \frac{\phi(z)}{(z-z_1)^2} \quad \rightarrow \quad \phi(z) = \frac{-2iz}{(z-z_2)^2}$$

So

$$\operatorname*{Res}_{z=z_1} f(z) = \phi'(z_1) = -\frac{ai}{2(a^2-1)^{3/2}}.$$

This means that

$$\int_C f(z)\,dz = 2\pi i\left(-\frac{ai}{2(\sqrt{a^2-1})^3}\right) = \frac{a\pi}{(\sqrt{a^2-1})^3}.$$

Therefore,

$$\int_0^\pi \frac{d\theta}{(a+\cos\theta)^2} = \frac{a\pi}{(\sqrt{a^2-1})^3} \quad (a > 1).$$