Exercise 6

Use residues to establish the following integration formula:

$$\int_0^{\pi} \sin^{2n}\theta\, d\theta = \frac{(2n)!}{2^{2n}(n!)^2}\pi \quad (n = 1, 2, \ldots).$$

Solution

Make the substitution,

$$\alpha = 2\theta \quad \rightarrow \quad \frac{\alpha}{2} = \theta$$

$$d\alpha = 2\, d\theta \quad \rightarrow \quad \frac{d\alpha}{2} = d\theta,$$

so that the limits of integration go from 0 to 2π.

$$\int_0^{2\pi} \sin^{2n}\frac{\alpha}{2}\left(\frac{d\alpha}{2}\right)$$

$$\frac{1}{2}\int_0^{2\pi}\left(\sin\frac{\alpha}{2}\right)^{2n} d\alpha$$

$$\frac{1}{2}\int_0^{2\pi}\left(\sqrt{\frac{1-\cos\alpha}{2}}\right)^{2n} d\alpha$$

$$\frac{1}{2}\int_0^{2\pi}\left(\frac{1-\cos\alpha}{2}\right)^{n} d\alpha$$

$$\frac{1}{2}\int_0^{2\pi}\left(\frac{1-\frac{e^{i\alpha}+e^{-i\alpha}}{2}}{2}\right)^{n} d\alpha$$

$$\frac{1}{2}\int_0^{2\pi}\left(\frac{2-e^{i\alpha}-e^{-i\alpha}}{2^2}\right)^{n} d\alpha$$

$$\frac{1}{2^{2n+1}}\int_0^{2\pi}\left(2-e^{i\alpha}-\frac{1}{e^{i\alpha}}\right)^{n} d\alpha$$

$$\frac{1}{2^{2n+1}}\int_0^{2\pi}\left[\frac{2e^{i\alpha}-(e^{i\alpha})^2-1}{e^{i\alpha}}\right]^{n} d\alpha$$

Make the following substitution.

$$z = e^{i\alpha}$$

$$dz = ie^{i\alpha}\, d\alpha = iz\, d\alpha \quad \rightarrow \quad \frac{dz}{iz} = d\alpha$$

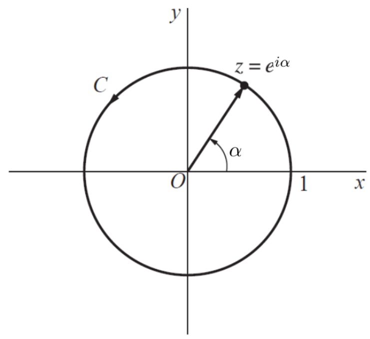

Figure 1: This figure illustrates the unit circle in the complex plane, where $z = x + iy$.

Doing so changes this integral into a counterclockwise contour integral over the unit circle in the complex plane.

$$\frac{1}{2^{2n+1}} \oint_C \left(\frac{2z - z^2 - 1}{z}\right)^n \left(\frac{dz}{iz}\right)$$

$$\frac{1}{i \cdot 2^{2n+1}} \oint_C \frac{(2z - z^2 - 1)^n}{z^{n+1}}\, dz$$

According to the Cauchy residue theorem, this closed contour integral is equal to $2\pi i$ times the sum of the residues at the enclosed singularities of the integrand. There's only one singular point to consider, namely $z = 0$.

$$\frac{1}{i \cdot 2^{2n+1}} \cdot 2\pi i \operatorname*{Res}_{z=0} \frac{(2z - z^2 - 1)^n}{z^{n+1}}$$

$$\frac{\pi}{2^{2n}} \operatorname*{Res}_{z=0} \frac{(2z - z^2 - 1)^n}{z^{n+1}}$$

Since $\phi(z) = (2z - z^2 - 1)^n$ is analytic and nonzero at 0, $z = 0$ is a pole of order $n + 1$, so the residue there can be calculated by

$$\frac{\pi}{2^{2n}} \frac{\phi^{[(n+1)-1]}(0)}{[(n+1) - 1]!}$$

$$\frac{\pi}{2^{2n}} \frac{\phi^{(n)}(0)}{n!}.$$

The goal now is to determine the nth derivative of $\phi(z)$.

$$n=1: \quad \phi^{(1)}(z) = 2(1-z) = 2\cdot 1(1-z)$$

$$n=2: \quad \phi^{(2)}(z) = 12(1-z)^2 = 4\cdot 3(1-z)^2 = \frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 1}(1-z)^2$$

$$n=3: \quad \phi^{(3)}(z) = 120(1-z)^3 = 6\cdot 5\cdot 4(1-z)^3 = \frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1}(1-z)^3$$

$$n=4: \quad \phi^{(4)}(z) = 1680(1-z)^4 = 8\cdot 7\cdot 6\cdot 5(1-z)^4 = \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{4\cdot 3\cdot 2\cdot 1}(1-z)^4$$

$$n=5: \quad \phi^{(5)}(z) = 30\,240(1-z)^5 = 10\cdot 9\cdot 8\cdot 7\cdot 6(1-z)^5 = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{5\cdot 4\cdot 3\cdot 2\cdot 1}(1-z)^5$$

$$\vdots$$

$$\phi^{(n)}(z) = \frac{(2n)!}{n!}(1-z)^n$$

So then

$$\phi^{(n)}(0) = \frac{(2n)!}{n!}.$$

Therefore,

$$\int_0^\pi \sin^{2n}\theta\,d\theta = \frac{\pi}{2^{2n}}\frac{\phi^{(n)}(0)}{n!} = \frac{\pi}{2^{2n}}\frac{1}{n!}\left[\frac{(2n)!}{n!}\right] = \frac{(2n)!}{2^{2n}(n!)^2}\pi, \quad n = 1, 2, \ldots.$$