Exercise 2

In each of the Exercises 1 through 3, use residues to find the inverse Laplace transform $f(t)$ corresponding to the given function $F(s)$. Do this in a formal way, without full justification,

$$F(s) = \frac{2s-2}{(s+1)(s^2+2s+5)}.$$

Ans. $f(t) = e^{-t}(\cos 2t + \sin 2t - 1)$.

Solution

Start off by finding the singularities of $F(s)$.

$$s^2 + 2s + 5 = 0 \quad \to \quad s = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

Hence, there are three singularities.

$$s_1 = -1 \quad s_2 = -1 + 2i \quad s_3 = -1 - 2i$$

The inverse Laplace transform is given by

$$f(t) = \sum_{n=1}^{3} \operatorname*{Res}_{s=s_n} [e^{st}F(s)].$$

We have

$$e^{st}F(s) = \frac{(2s-2)e^{st}}{(s-s_1)(s-s_2)(s-s_3)}.$$

Since all the factors in the denominator have multiplicity 1, s_1, s_2, and s_3 are simple poles, so the residues are of the form $\phi_n(s_n)$.

$$\text{Let } \phi_1(s) = \frac{(2s-2)e^{st}}{(s-s_2)(s-s_3)}. \quad \text{Then } \operatorname*{Res}_{s=s_1} [e^{st}F(s)] = \operatorname*{Res}_{s=s_1} \frac{\phi_1(s)}{s-s_1} = \phi_1(s_1) = -e^{-t}.$$

$$\text{Let } \phi_2(s) = \frac{(2s-2)e^{st}}{(s-s_1)(s-s_3)}. \quad \text{Then } \operatorname*{Res}_{s=s_2} [e^{st}F(s)] = \operatorname*{Res}_{s=s_2} \frac{\phi_2(s)}{s-s_2} = \phi_2(s_2) = \frac{1}{2}(1-i)e^{(-1+2i)t}.$$

$$\text{Let } \phi_3(s) = \frac{(2s-2)e^{st}}{(s-s_1)(s-s_2)}. \quad \text{Then } \operatorname*{Res}_{s=s_3} [e^{st}F(s)] = \operatorname*{Res}_{s=s_3} \frac{\phi_3(s)}{s-s_3} = \phi_3(s_3) = \frac{1}{2}(1+i)e^{(-1-2i)t}.$$

Summing the residues we obtain $f(t)$, the inverse Laplace transform of $F(s)$.

$$\begin{aligned} f(t) = \sum_{n=1}^{3} \operatorname*{Res}_{s=s_n} [e^{st}F(s)] &= -e^{-t} + \frac{1}{2}(1-i)e^{(-1+2i)t} + \frac{1}{2}(1+i)e^{(-1-2i)t} \\ &= e^{-t}\left(-1 + \frac{e^{2it}+e^{-2it}}{2} + \frac{e^{2it}-e^{-2it}}{2i}\right) \end{aligned}$$

Therefore,

$$f(t) = e^{-t}(-1 + \cos 2t + \sin 2t).$$