Exercise 4

Follow the steps below to find $f(t)$ when

$$F(s) = \frac{1}{s^2} - \frac{1}{s\sinh s}.$$

Start with the observation that the isolated singularities of $F(s)$ are

$$s_0 = 0, \quad s_n = n\pi i, \quad \overline{s_n} = -n\pi i \qquad (n = 1, 2, \ldots).$$

(*a*) Use the Laurent series found in Exercise 5, Sec. 73, to show that the function $e^{st}F(s)$ has a removable singularity at $s = s_0$, with residue 0.

(*b*) Use Theorem 2 in Sec. 83 to show that

$$\operatorname*{Res}_{s=s_n}\left[e^{st}F(s)\right] = \frac{(-1)^n i\exp(in\pi t)}{n\pi}$$

and

$$\operatorname*{Res}_{s=\overline{s_n}}\left[e^{st}F(s)\right] = \frac{-(-1)^n i\exp(-in\pi t)}{n\pi}.$$

(*c*) Show how it follows from parts (*a*) and (*b*), together with series (7), Sec. 95, that

$$f(t) = \sum_{n=1}^{\infty}\left\{\operatorname*{Res}_{s=s_n}\left[e^{st}F(s)\right] + \operatorname*{Res}_{s=\overline{s_n}}\left[e^{st}F(s)\right]\right\} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi t.$$

Solution

Write the function so that it has one term.

$$\begin{aligned} F(s) &= \frac{1}{s^2} - \frac{1}{s\sinh s} \\ &= \frac{\sinh s}{s^2\sinh s} - \frac{s}{s^2\sinh s} \\ &= \frac{\sinh s - s}{s^2\sinh s} \end{aligned}$$

Find the singularities—they occur where the denominator is equal to zero.

$$s^2 = 0 \qquad \text{or} \qquad \sinh s = 0$$

Use the identity $\sinh s = -i\sin is$.

$$\begin{aligned} s = 0 \qquad \text{or} \qquad -i\sin is &= 0 \\ \sin is &= 0 \\ is &= n\pi, \qquad n = 0, \pm 1, \pm 2, \ldots \\ s &= -in\pi, \end{aligned}$$

Hence, there are an infinite number of singularities. To be consistent with Churchill and Brown's notation, let

$$s_0 = 0, \quad s_n = in\pi, \quad \overline{s_n} = -in\pi, \quad n = 1, 2, \ldots.$$

Part (a)

According to Exercise 5, Sec. 73 on page 225,

$$\frac{1}{z^2 \sinh z} = \frac{1}{z^3} - \frac{1}{6} \cdot \frac{1}{z} + \frac{7}{360} z + \cdots .$$

The aim here is to determine

$$\operatorname*{Res}_{s=s_0} [e^{st} F(s)]$$

by writing out the Laurent series of $e^{st}F(s)$ and inspecting the coefficient of $1/s$.

$$\begin{aligned} e^{st}F(s) &= \frac{\sinh s - s}{s^2 \sinh s} e^{st} \\ &= (\sinh s - s) \frac{1}{s^2 \sinh s} e^{st}. \end{aligned}$$

In order to find the residue at $s = s_0 = 0$, expand each of the functions about $s = 0$.

$$\begin{aligned} &= \left(s + \frac{s^3}{6} + \frac{s^5}{120} + \cdots - s\right)\left(\frac{1}{s^3} - \frac{1}{6s} + \frac{7}{360}s + \cdots\right)\left(1 + ts + \frac{t^2 s^2}{2} + \cdots\right) \\ &= \left(\frac{s^3}{6} + \frac{s^5}{120} + \cdots\right)\left(\frac{1}{s^3} - \frac{1}{6s} + \frac{7}{360}s + \cdots\right)\left(1 + ts + \frac{t^2 s^2}{2} + \cdots\right) \end{aligned}$$

Proceed with the multiplication.

$$= \frac{1}{6} + \frac{t}{6}s + \left(-\frac{7}{360} + \frac{t^2}{12}\right)s^2 + \left(-\frac{7t}{360} + \frac{t^3}{36}\right)s^3 + \cdots$$

The coefficient of $1/s$ is the residue at $s = 0$. Thus,

$$\operatorname*{Res}_{s=s_0} [e^{st} F(s)] = 0.$$

Part (b)

Theorem 2 in Sec. 83 reads as follows: Assume there are two analytic functions, p and q, at a point z_0. If

$$p(z_0) \neq 0, \quad q(z_0) = 0, \quad \text{and} \quad q'(z_0) \neq 0,$$

then z_0 is a simple pole of the quotient $p(z)/q(z)$ and

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

In order to evaluate the residue of $e^{st}F(s)$ at s_n and $\overline{s_n}$, choose

$$p(s) = e^{st}(\sinh s - s) \quad \text{and} \quad q(s) = s^2 \sinh s.$$

Then

$$q'(s) = s(s \cosh s + 2 \sinh s).$$

Check that the hypotheses of the theorem are satisfied for both $s_n = in\pi$ and $\overline{s_n} = -in\pi$.

$$p(in\pi) = e^{(in\pi)t}(\sinh in\pi - in\pi) = e^{in\pi t}(i\sin n\pi - in\pi) = e^{in\pi t}(-in\pi) \neq 0$$

$$p(-in\pi) = e^{(-in\pi)t}[\sinh(-in\pi) - (-in\pi)] = e^{-in\pi t}(-i\sin n\pi + in\pi) = e^{-in\pi t}(in\pi) \neq 0$$

$$q(in\pi) = (in\pi)^2 \sinh(in\pi) = -n^2\pi^2[i\sin(n\pi)] = 0$$

$$q(-in\pi) = (-in\pi)^2 \sinh(-in\pi) = -n^2\pi^2[-i\sin(n\pi)] = 0$$

$$q'(in\pi) = (in\pi)(in\pi\cosh in\pi + 2\sinh in\pi) = in\pi(in\pi\cos n\pi + 2i\sin n\pi) = (in\pi)^2\cos n\pi \neq 0$$

$$q'(-in\pi) = (-in\pi)[(-in\pi)\cosh(-in\pi) + 2\sinh(-in\pi)] = -in\pi(-in\pi\cos n\pi - 2i\sin n\pi) = (-in\pi)^2\cos n\pi \neq 0$$

Consequently, $s_n = in\pi$ and $\overline{s_n} = -in\pi$ are simple poles of $e^{st}F(s)$, and

$$\operatorname*{Res}_{s=s_n} e^{st}F(s) = \frac{p(s_n)}{q'(s_n)} = \frac{p(in\pi)}{q'(in\pi)} = \frac{e^{in\pi t}(-in\pi)}{(in\pi)^2\cos n\pi} = -\frac{e^{in\pi t}}{in\pi(-1)^n} \times \frac{i(-1)^n}{i(-1)^n} = \frac{(-1)^n i\exp(in\pi t)}{n\pi}$$

$$\operatorname*{Res}_{s=\overline{s_n}} e^{st}F(s) = \frac{p(\overline{s_n})}{q'(\overline{s_n})} = \frac{p(-in\pi)}{q'(-in\pi)} = \frac{e^{-in\pi t}(in\pi)}{(-in\pi)^2\cos n\pi} = \frac{e^{-in\pi t}}{in\pi(-1)^n} \times \frac{i(-1)^n}{i(-1)^n} = \frac{-(-1)^n i\exp(-in\pi t)}{n\pi}$$

by Theorem 2 in Sec. 83.

Part (c)

According to Equation (7) in Sec. 95 on page 296, the inverse Laplace transform is obtained by summing the residues of $e^{st}F(s)$ at each of the singularities. Note that each singularity lies on the complex axis, which is to the left of any vertical line on the right half of the complex plane.

$$\begin{aligned}
f(t) &= \operatorname*{Res}_{s=s_0} e^{st}F(s) + \sum_{n=1}^{\infty} \operatorname*{Res}_{s=s_n} e^{st}F(s) + \sum_{n=1}^{\infty} \operatorname*{Res}_{s=\overline{s_n}} e^{st}F(s) \\
&= 0 + \sum_{n=1}^{\infty} \frac{(-1)^n i\exp(in\pi t)}{n\pi} + \sum_{n=1}^{\infty} \frac{-(-1)^n i\exp(-in\pi t)}{n\pi} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n i}{n\pi}(e^{in\pi t} - e^{-in\pi t}) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n i}{n\pi}(2i\sin n\pi t) \\
&= \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin n\pi t
\end{aligned}$$