Exercise 5

By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that

(a) $i(1-\sqrt{3}i)(\sqrt{3}+i) = 2(1+\sqrt{3}i)$; (b) $5i/(2+i) = 1+2i$;

(c) $(\sqrt{3}+i)^6 = -64$; (d) $(1+\sqrt{3}i)^{-10} = 2^{-11}(-1+\sqrt{3}i)$.

Solution

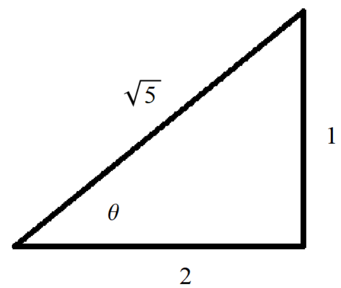
Part (a)

$$\begin{aligned}
i(1-\sqrt{3}i)(\sqrt{3}+i) &= \left(e^{i\pi/2}\right)\left[\sqrt{1^2+(-\sqrt{3})^2}\exp\left(i\tan^{-1}\frac{-\sqrt{3}}{1}\right)\right]\left[\sqrt{(\sqrt{3})^2+1^2}\exp\left(i\tan^{-1}\frac{1}{\sqrt{3}}\right)\right] \\
&= e^{i\pi/2}(2e^{-i\pi/3})(2e^{i\pi/6}) \\
&= 4e^{i(\pi/2-\pi/3+\pi/6)} \\
&= 4e^{i\pi/3} \\
&= 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \\
&= 4\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\
&= 2(1+\sqrt{3}i)
\end{aligned}$$

Part (b)

$$\begin{aligned}
\frac{5i}{2+i} &= \frac{5e^{i\pi/2}}{\sqrt{2^2+1^2}\exp\left(i\tan^{-1}\frac{1}{2}\right)} \\
&= \sqrt{5}\exp\left[i\left(\frac{\pi}{2} - \tan^{-1}\frac{1}{2}\right)\right] \\
&= \sqrt{5}\left[\cos\left(\frac{\pi}{2} - \tan^{-1}\frac{1}{2}\right) + i\sin\left(\frac{\pi}{2} - \tan^{-1}\frac{1}{2}\right)\right] \\
&= \sqrt{5}\left(\underbrace{\cos\frac{\pi}{2}}_{=0}\cos\tan^{-1}\frac{1}{2} + \underbrace{\sin\frac{\pi}{2}}_{=1}\sin\tan^{-1}\frac{1}{2}\right) + i\left(\underbrace{\sin\frac{\pi}{2}}_{=1}\cos\tan^{-1}\frac{1}{2} - \underbrace{\cos\frac{\pi}{2}}_{=0}\sin\tan^{-1}\frac{1}{2}\right) \\
&= \sqrt{5}\left(\sin\tan^{-1}\frac{1}{2} + i\cos\tan^{-1}\frac{1}{2}\right)
\end{aligned}$$

Let $\theta = \tan^{-1}(1/2)$ and draw the implied right triangle to determine the quantities in parentheses.



So then

$$\begin{aligned}\frac{5i}{2+i} &= \sqrt{5}(\sin\theta + i\cos\theta)\\ &= \sqrt{5}\left(\frac{1}{\sqrt{5}} + i\frac{2}{\sqrt{5}}\right)\\ &= 1 + 2i.\end{aligned}$$

Part (c)

$$\begin{aligned}(\sqrt{3}+i)^6 &= \left\{\sqrt{(\sqrt{3})^2 + 1^2}\exp\left[i\left(\tan^{-1}\frac{1}{\sqrt{3}}\right)\right]\right\}^6\\ &= \left[2e^{i(\pi/6)}\right]^6\\ &= 2^6 e^{i\pi}\\ &= 64(\cos\pi + i\sin\pi)\\ &= 64(-1)\\ &= -64\end{aligned}$$

Part (d)

$$\begin{aligned}(1+\sqrt{3}i)^{-10} &= \left[\sqrt{1^2 + (\sqrt{3})^2}\exp\left(i\tan^{-1}\frac{\sqrt{3}}{1}\right)\right]^{-10}\\ &= (2e^{i\pi/3})^{-10}\\ &= 2^{-10}e^{-10i\pi/3}\\ &= 2^{-10}\left(\cos\frac{10\pi}{3} - i\sin\frac{10\pi}{3}\right)\\ &= 2^{-10}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\\ &= 2^{-11}(-1+\sqrt{3}i)\end{aligned}$$