Exercise 8

Prove that two nonzero complex numbers z_1 and z_2 have the same moduli if and only if there are complex numbers c_1 and c_2 such that $z_1 = c_1c_2$ and $z_2 = c_1\overline{c_2}$.

Suggestion: Note that

$$\exp\left(i\frac{\theta_1+\theta_2}{2}\right)\exp\left(i\frac{\theta_1-\theta_2}{2}\right) = \exp(i\theta_1)$$

and

$$\exp\left(i\frac{\theta_1+\theta_2}{2}\right)\overline{\exp\left(i\frac{\theta_1-\theta_2}{2}\right)} = \exp(i\theta_2).$$

Solution

Suppose that there are two nonzero complex numbers, z_1 and z_2,

$$z_1 = r_1e^{i\theta_1} \qquad z_2 = r_2e^{i\theta_2}$$

and that they have the same moduli.

$$|z_1| = |z_2| \quad \rightarrow \quad |r_1e^{i\theta_1}| = |r_2e^{i\theta_2}| \quad \rightarrow \quad r_1 = r_2$$

If we choose c_1 and c_2 to be

$$\begin{aligned} c_1 &= r_1\exp\left(i\frac{\theta_1+\theta_2}{2}\right) \\ c_2 &= \exp\left(i\frac{\theta_1-\theta_2}{2}\right), \end{aligned}$$

then

$$\begin{aligned} z_1 = c_1c_2 &= r_1\exp\left(i\frac{\theta_1+\theta_2}{2}\right)\exp\left(i\frac{\theta_1-\theta_2}{2}\right) = r_1e^{i\theta_1} \\ z_2 = c_1\overline{c_2} &= r_1\exp\left(i\frac{\theta_1+\theta_2}{2}\right)\overline{\exp\left(i\frac{\theta_1-\theta_2}{2}\right)} = r_1e^{i\theta_2} = r_2e^{i\theta_2}. \end{aligned}$$

The first part of the proof is complete. Suppose now that there exist complex numbers, c_1 and c_2, such that $z_1 = c_1c_2$ and $z_2 = c_1\overline{c_2}$. The aim is to show that z_1 and z_2 have the same moduli. If we have

$$\begin{aligned} c_1 &= a_1 + ib_1 \\ c_2 &= a_2 + ib_2, \end{aligned}$$

then

$$\begin{aligned} z_1 = c_1c_2 &= (a_1+ib_1)(a_2+ib_2) = a_1a_2 - b_1b_2 + i(a_1b_2 + a_2b_1) \\ z_2 = c_1\overline{c_2} &= (a_1+ib_1)(a_2-ib_2) = a_1a_2 + b_1b_2 + i(-a_1b_2 + a_2b_1), \end{aligned}$$

and the magnitudes are

$$\begin{aligned} |z_1| &= \sqrt{(a_1a_2-b_1b_2)^2 + (a_1b_2+a_2b_1)^2} = \sqrt{a_1^2a_2^2 + a_2^2b_1^2 + a_1^2b_2^2 + b_1^2b_2^2} \\ |z_2| &= \sqrt{(a_1a_2+b_1b_2)^2 + (-a_1b_2+a_2b_1)^2} = \sqrt{a_1^2a_2^2 + a_2^2b_1^2 + a_1^2b_2^2 + b_1^2b_2^2} \end{aligned}$$

in fact equal. Therefore, two nonzero complex numbers, z_1 and z_2, have the same moduli if and only if there are complex numbers, c_1 and c_2, such that $z_1 = c_1c_2$ and $z_2 = c_1\overline{c_2}$.