Exercise 1

Classify each of the partial differential equations below as either hyperbolic, parabolic, or elliptic, determine the characteristics, and transform the equations to canonical form:

(a) $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$

(b) $2u_{xx} - 3u_{xy} + u_{yy} = y$

(c) $yu_{xx} + (x + y)u_{xy} + xu_{yy} = 0$

(d) $u_{xx} + yu_{yy} = 0$

(e) $yu_{xx} - 2u_{xy} + e^x u_{yy} + x^2 u_x - u = 0$

(f) $u_{xx} + xu_{yy} = 0$

(g) $x^2 u_{xx} + 4yu_{xy} + u_{yy} + 2u_x = 0$

(h) $3yu_{xx} - xu_{yy} = 0$

(i) $u_{xx} + 2xu_{xy} + a^2 u_{yy} + u = 5$

(j) $y^2 u_{xx} + x^2 u_{yy} = 0$

Solution

Part (a)

$4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$

Comparing this equation with the general form of a second-order PDE,
$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 4$, $B = 5$, $C = 1$, $D = 1$, $E = 1$, $F = 0$, and $G = 2$. The characteristic equations of this PDE are given by

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right) \\
\frac{dy}{dx} &= \frac{1}{8}\left(5 \pm \sqrt{25 - 16}\right) \\
\frac{dy}{dx} &= \frac{1}{8}(5 \pm 3) \\
\frac{dy}{dx} &= 1 \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{4}.
\end{aligned}$$

Note that the discriminant, $B^2 - 4AC = 25 - 16 = 9$, is greater than 0, which means that the PDE is **hyperbolic**. Therefore, the solutions to the ordinary differential equations are two real and distinct families of characteristic curves in the xy-plane.

$$y = x + C_1 \quad \text{or} \quad y = \frac{1}{4}x + C_2.$$

Solve for the constants of integration.

$$\begin{aligned}
C_1 &= y - x \\
C_2 &= y - \frac{1}{4}x
\end{aligned}$$

Make the change of variables, $\xi = y - x$ and $\eta = y - \frac{1}{4}x$, so that the PDE takes the simplest form. Use the chain rule to write the derivatives in terms of these new variables.

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi}(-1) + \frac{\partial u}{\partial \eta}\left(-\frac{1}{4}\right) = -u_\xi - \frac{1}{4}u_\eta \\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi}(1) + \frac{\partial u}{\partial \eta}(1) = u_\xi + u_\eta \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial u}{\partial x}\right) = \left(-\frac{\partial}{\partial \xi} - \frac{1}{4}\frac{\partial}{\partial \eta}\right)\left(-\frac{\partial u}{\partial \xi} - \frac{1}{4}\frac{\partial u}{\partial \eta}\right) = u_{\xi\xi} + \frac{1}{2}u_{\xi\eta} + \frac{1}{16}u_{\eta\eta} \\
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(\frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial u}{\partial y}\right) = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \\
\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial u}{\partial y}\right) = \left(-\frac{\partial}{\partial \xi} - \frac{1}{4}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) = -u_{\xi\xi} - \frac{5}{4}u_{\xi\eta} - \frac{1}{4}u_{\eta\eta}
\end{aligned}$$

Substitute these formulas into the PDE.

$$\begin{aligned}
4\left(u_{\xi\xi} + \frac{1}{2}u_{\xi\eta} + \frac{1}{16}u_{\eta\eta}\right) + 5\left(-u_{\xi\xi} - \frac{5}{4}u_{\xi\eta} - \frac{1}{4}u_{\eta\eta}\right) + (u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) \\
+ \left(-u_\xi - \frac{1}{4}u_\eta\right) + (u_\xi + u_\eta) = 2
\end{aligned}$$

Simplify the left side.

$$-\frac{9}{4}u_{\xi\eta} + \frac{3}{4}u_\eta = 2$$

Solve for $u_{\xi\eta}$.

$$u_{\xi\eta} = \frac{1}{3}u_\eta - \frac{8}{9}$$

This is the first canonical form of the hyperbolic PDE. Make the additional change of variables, $\alpha = \xi + \eta$ and $\beta = \xi - \eta$. Use the chain rule again to write the derivatives in terms of these new variables.

$$\begin{aligned}
\frac{\partial u}{\partial \xi} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial \xi} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial \xi} = \frac{\partial u}{\partial \alpha}(1) + \frac{\partial u}{\partial \beta}(1) = u_\alpha + u_\beta \\
\frac{\partial u}{\partial \eta} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial \eta} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial \eta} = \frac{\partial u}{\partial \alpha}(1) + \frac{\partial u}{\partial \beta}(-1) = u_\alpha - u_\beta \\
\frac{\partial^2 u}{\partial \xi \partial \eta} &= \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta}\right) = \left(\frac{\partial \alpha}{\partial \xi}\frac{\partial}{\partial \alpha} + \frac{\partial \beta}{\partial \xi}\frac{\partial}{\partial \beta}\right)\left(\frac{\partial u}{\partial \eta}\right) = \left(\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}\right)(u_\alpha - u_\beta) = u_{\alpha\alpha} - u_{\beta\beta}
\end{aligned}$$

Substitute these formulas into the first canonical form.

$$u_{\alpha\alpha} - u_{\beta\beta} = \frac{1}{3}(u_\alpha - u_\beta) - \frac{8}{9}$$

This is the second canonical form of the hyperbolic PDE.

Part (b)

$2u_{xx} - 3u_{xy} + u_{yy} = y$

Comparing this equation with the general form of a second-order PDE,
$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 2$, $B = -3$, $C = 1$, $D = 0$, $E = 0$, $F = 0$, and $G = y$. The characteristic equations of this PDE are given by

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)\\ \frac{dy}{dx} &= \frac{1}{4}\left(-3 \pm \sqrt{9-8}\right)\\ \frac{dy}{dx} &= \frac{1}{4}(-3 \pm 1)\\ \frac{dy}{dx} &= -1 \quad \text{or} \quad \frac{dy}{dx} = -\frac{1}{2}.\end{aligned}$$

Note that the discriminant, $B^2 - 4AC = 9 - 8 = 1$, is greater than 0, which means that the PDE is **hyperbolic**. Therefore, the solutions to the ordinary differential equations are two real and distinct families of characteristic curves in the xy-plane.

$$y = -x + C_1 \quad \text{or} \quad y = -\frac{1}{2}x + C_2$$

Solve for the constants of integration.

$$\begin{aligned}C_1 &= y + x\\ C_2 &= y + \frac{1}{2}x\end{aligned}$$

Make the change of variables, $\xi = y + x$ and $\eta = y + \frac{1}{2}x$, so that the PDE takes the simplest form. Use the chain rule to write the derivatives in terms of these new variables.

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi}(1) + \frac{\partial u}{\partial \eta}\left(\frac{1}{2}\right) = u_\xi + \frac{1}{2}u_\eta\\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi}(1) + \frac{\partial u}{\partial \eta}(1) = u_\xi + u_\eta\\ \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial}{\partial \xi} + \frac{1}{2}\frac{\partial}{\partial \eta}\right)\left(u_\xi + \frac{1}{2}u_\eta\right) = u_{\xi\xi} + u_{\xi\eta} + \frac{1}{4}u_{\eta\eta}\\ \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(\frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial u}{\partial y}\right) = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)(u_\xi + u_\eta) = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\\ \frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial u}{\partial y}\right) = \left(\frac{\partial}{\partial \xi} + \frac{1}{2}\frac{\partial}{\partial \eta}\right)(u_\xi + u_\eta) = u_{\xi\xi} + \frac{3}{2}u_{\xi\eta} + \frac{1}{2}u_{\eta\eta}\end{aligned}$$

Additionally, solving the change of variables for x and y yields $x = 2(\xi - \eta)$ and $y = 2\eta - \xi$. Substitute these formulas into the PDE.

$$2\left(u_{\xi\xi} + u_{\xi\eta} + \frac{1}{4}u_{\eta\eta}\right) - 3\left(u_{\xi\xi} + \frac{3}{2}u_{\xi\eta} + \frac{1}{2}u_{\eta\eta}\right) + (u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) = 2\eta - \xi$$

Simplify the left side.

$$-\frac{1}{2}u_{\xi\eta} = 2\eta - \xi.$$

Solving for $u_{\xi\eta}$ gives

$$u_{\xi\eta} = 2(\xi - 2\eta).$$

This is the first canonical form of the hyperbolic PDE. Make the additional change of variables, $\alpha = \xi + \eta$ and $\beta = \xi - \eta$. Use the chain rule again to write the derivatives in terms of these new variables.

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial \xi} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial \xi} = \frac{\partial u}{\partial \alpha}(1) + \frac{\partial u}{\partial \beta}(1) = u_\alpha + u_\beta$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial \eta} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial \eta} = \frac{\partial u}{\partial \alpha}(1) + \frac{\partial u}{\partial \beta}(-1) = u_\alpha - u_\beta$$

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta}\right) = \left(\frac{\partial \alpha}{\partial \xi}\frac{\partial}{\partial \alpha} + \frac{\partial \beta}{\partial \xi}\frac{\partial}{\partial \beta}\right)\left(\frac{\partial u}{\partial \eta}\right) = \left(\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}\right)(u_\alpha - u_\beta) = u_{\alpha\alpha} - u_{\beta\beta}$$

Additionally, solving the change of variables for ξ and η yields $\xi = \alpha/2 + \beta/2$ and $\eta = \alpha/2 - \beta/2$. Substitute these formulas into the first canonical form.

$$u_{\alpha\alpha} - u_{\beta\beta} = 2\left[\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) - 2\left(\frac{\alpha}{2} - \frac{\beta}{2}\right)\right]$$

$$u_{\alpha\alpha} - u_{\beta\beta} = 3\beta - \alpha$$

This is the second canonical form of the hyperbolic PDE.

Part (c)

$yu_{xx} + (x+y)u_{xy} + xu_{yy} = 0$

Comparing this equation with the general form of a second-order PDE, $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = y$, $B = x + y$, $C = x$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right) \\
\frac{dy}{dx} &= \frac{1}{2y}\left(x + y \pm \sqrt{(x+y)^2 - 4xy}\right) \\
\frac{dy}{dx} &= \frac{1}{2y}\left(x + y \pm \sqrt{(x-y)^2}\right) \\
\frac{dy}{dx} &= \frac{1}{2y}\left(x + y \pm |x-y|\right).
\end{aligned}$$

Note that the discriminant, $B^2 - 4AC = (x-y)^2$, is greater than 0 for all x and y, which means that the PDE is **hyperbolic**. Therefore, the solutions to the ordinary differential equations are two real and distinct families of characteristic curves in the xy-plane.

$$\begin{aligned}
x > y \Rightarrow |x-y| = x-y: &\quad \frac{dy}{dx} = \frac{1}{2y}(2x) = \frac{x}{y}, \quad \frac{dy}{dx} = \frac{1}{2y}(2y) = 1 \\
x < y \Rightarrow |x-y| = y-x: &\quad \frac{dy}{dx} = \frac{1}{2y}(2y) = 1, \quad \frac{dy}{dx} = \frac{1}{2y}(2x) = \frac{x}{y}
\end{aligned}$$

The two characteristic equations are the same regardless of whether x is greater than y or not:

$$\frac{dy}{dx} = \frac{x}{y} \quad \text{or} \quad \frac{dy}{dx} = 1.$$

Integrate these equations.

$$\frac{y^2}{2} = \frac{x^2}{2} + C_1 \quad \text{or} \quad y = x + C_2$$

Solve for the constants of integration (or any convenient multiple thereof).

$$\begin{aligned}
2C_1 &= y^2 - x^2 \\
C_2 &= y - x
\end{aligned}$$

Make the change of variables, $\xi = y^2 - x^2$ and $\eta = y - x$, so that the PDE takes the simplest form. Solving these equations for x and y yields

$$x = \frac{\xi - \eta^2}{2\eta} \quad \text{and} \quad y = \frac{\xi + \eta^2}{2\eta}.$$

Use the chain rule to write the derivatives in terms of ξ and η.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi}(-2x) + \frac{\partial u}{\partial \eta}(-1) = -2xu_\xi - u_\eta = \frac{\eta^2 - \xi}{\eta}u_\xi - u_\eta$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi}(2y) + \frac{\partial u}{\partial \eta}(1) = 2yu_\xi + u_\eta = \frac{\xi + \eta^2}{\eta}u_\xi + u_\eta$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial u}{\partial x}\right) &= \left(\frac{\eta^2 - \xi}{\eta}\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right)\left(\frac{\eta^2 - \xi}{\eta}u_\xi - u_\eta\right) \\
&= \frac{(\eta^2 - \xi)^2}{\eta^2}u_{\xi\xi} - \frac{2}{\eta}(\eta^2 - \xi)u_{\xi\eta} - 2u_\xi + u_{\eta\eta}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(\frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial u}{\partial y}\right) &= \left(\frac{\xi + \eta^2}{\eta}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\xi + \eta^2}{\eta}u_\xi + u_\eta\right) \\
&= \frac{(\xi + \eta^2)^2}{\eta^2}u_{\xi\xi} + \frac{2}{\eta}(\eta^2 + \xi)u_{\xi\eta} + 2u_\xi + u_{\eta\eta}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial u}{\partial y}\right) &= \left(\frac{\eta^2 - \xi}{\eta}\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right)\left(\frac{\xi + \eta^2}{\eta}u_\xi + u_\eta\right) \\
&= \frac{\eta^4 - \xi^2}{\eta^2}u_{\xi\xi} - \frac{2\xi}{\eta}u_{\xi\eta} - u_{\eta\eta}
\end{aligned}$$

Substitute these formulas into the PDE.

$$yu_{xx} + (x + y)u_{xy} + xu_{yy} = 0$$

$$\begin{aligned}
\left(\frac{\xi + \eta^2}{2\eta}\right)&\left[\frac{(\eta^2 - \xi)^2}{\eta^2}u_{\xi\xi} - \frac{2}{\eta}(\eta^2 - \xi)u_{\xi\eta} - 2u_\xi + u_{\eta\eta}\right] \\
&+ \left(\frac{\xi - \eta^2}{2\eta} + \frac{\xi + \eta^2}{2\eta}\right)\left[\frac{\eta^4 - \xi^2}{\eta^2}u_{\xi\xi} - \frac{2\xi}{\eta}u_{\xi\eta} - u_{\eta\eta}\right] \\
&+ \left(\frac{\xi - \eta^2}{2\eta}\right)\left[\frac{(\xi + \eta^2)^2}{\eta^2}u_{\xi\xi} + \frac{2}{\eta}(\eta^2 + \xi)u_{\xi\eta} + 2u_\xi + u_{\eta\eta}\right] = 0
\end{aligned}$$

Simplify the left side.

$$-2\eta^2 u_{\xi\eta} - 2\eta u_\xi = 0$$

Solve for $u_{\xi\eta}$.

$$u_{\xi\eta} = -\frac{1}{\eta}u_\xi$$

This is the first canonical form of the hyperbolic PDE. Make the additional change of variables, $\alpha = \xi + \eta$ and $\beta = \xi - \eta$. Use the chain rule again to write the derivatives in terms of these new variables.

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial \xi} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial \xi} = \frac{\partial u}{\partial \alpha}(1) + \frac{\partial u}{\partial \beta}(1) = u_\alpha + u_\beta$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial \eta} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial \eta} = \frac{\partial u}{\partial \alpha}(1) + \frac{\partial u}{\partial \beta}(-1) = u_\alpha - u_\beta$$

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta}\right) = \left(\frac{\partial \alpha}{\partial \xi}\frac{\partial}{\partial \alpha} + \frac{\partial \beta}{\partial \xi}\frac{\partial}{\partial \beta}\right)\left(\frac{\partial u}{\partial \eta}\right) = \left(\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}\right)(u_\alpha - u_\beta) = u_{\alpha\alpha} - u_{\beta\beta}$$

Additionally, solving the change of variables for ξ and η yields $\xi = \alpha/2 + \beta/2$ and $\eta = \alpha/2 - \beta/2$. Substitute these formulas into the first canonical form.

$$u_{\alpha\alpha} - u_{\beta\beta} = -\frac{1}{\frac{\alpha}{2} - \frac{\beta}{2}}(u_\alpha + u_\beta)$$

$$u_{\alpha\alpha} - u_{\beta\beta} = \frac{2}{\beta - \alpha}(u_\alpha + u_\beta)$$

This is the second canonical form of the hyperbolic PDE.

Part (d)

$u_{xx} + yu_{yy} = 0$

Comparing this equation with the general form of a second-order PDE, $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 1$, $B = 0$, $C = y$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right) \\ \frac{dy}{dx} &= \frac{1}{2}\left(\pm\sqrt{-4y}\right) \\ \frac{dy}{dx} &= \pm\sqrt{-y}. \end{aligned}$$

Note that the discriminant, $B^2 - 4AC = -4y$, can be positive, zero, or negative, depending on whether $y < 0$, $y = 0$, or $y > 0$, respectively. That is,

$$\text{The PDE is } \begin{cases} \text{hyperbolic} & \text{if } y < 0. \\ \text{parabolic} & \text{if } y = 0. \\ \text{elliptic} & \text{if } y > 0. \end{cases}$$

Let us consider each case individually.

Case I: The PDE is hyperbolic ($y < 0$)

The ordinary differential equations yield one real family of characteristic curves in the xy-plane. Separating variables and integrating both sides of the characteristic equations, we find that

$$2\sqrt{-y} = \pm x + C_0.$$

Solving for y, the characteristic curves are $y(x) = -\frac{1}{4}(x \pm C_0)^2$. Solving for the constant of integration (or any convenient multiple thereof),

$$\begin{aligned} &\text{Working with } -x: \quad && +C_0 = x + 2\sqrt{-y} = \phi(x, y) \\ &\text{Working with } +x: \quad && -C_0 = x - 2\sqrt{-y} = \psi(x, y). \end{aligned}$$

Now we make the change of variables, $\xi = \phi(x, y) = x + 2\sqrt{-y}$ and $\eta = \psi(x, y) = x - 2\sqrt{-y}$, so that the PDE takes the simplest form. Solving these two equations for x and $2\sqrt{-y}$ gives $x = (\xi + \eta)/2$ and $2\sqrt{-y} = (\xi - \eta)/2$. With these new variables the PDE becomes

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*,$$

where, using the chain rule, (see page 11 of the textbook for details)

$$\begin{aligned} A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\ B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\ C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\ D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\ E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\ F^* &= F \\ G^* &= G. \end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = 4$, $C^* = 0$, $D^* = \frac{1}{2\sqrt{-y}} = \frac{2}{\xi-\eta}$, $E^* = -\frac{1}{2\sqrt{-y}} = -\frac{2}{\xi-\eta}$, $F^* = 0$, and $G^* = 0$. Thus, the PDE simplifies to

$$4u_{\xi\eta} + \frac{2}{\xi-\eta}u_\xi - \frac{2}{\xi-\eta}u_\eta = 0.$$

Solving for $u_{\xi\eta}$ gives

$$u_{\xi\eta} = -\frac{1}{2(\xi-\eta)}(u_\xi - u_\eta).$$

This is the first canonical form of the hyperbolic PDE. If we make the additional change of variables, $\alpha = \xi + \eta$ and $\beta = \xi - \eta$, then the chain rule gives $u_{\xi\eta} = u_{\alpha\alpha} - u_{\beta\beta}$, $u_\xi = u_\alpha + u_\beta$, and $u_\eta = u_\alpha - u_\beta$. The PDE then becomes

$$u_{\alpha\alpha} - u_{\beta\beta} = -\frac{1}{\beta}u_\beta.$$

This is the second canonical form of the hyperbolic PDE.

Case II: The PDE is parabolic $(y = 0)$

Substituting $y = 0$ into the PDE reduces it immediately to the canonical form of a parabolic equation, $u_{xx} = 0$. The characteristic equation is given by

$$\frac{dy}{dx} = 0.$$

Solving this equation for y gives $y(x) = D$, where D is an arbitrary constant. The characteristic curves in the xy-plane are lines parallel to the x-axis.

Case III: The PDE is elliptic $(y > 0)$

Since the discriminant is negative for $y > 0$, the characteristic equations have no real solutions. This means that the family of characteristic curves lies in the complex plane:

$$\frac{dy}{dx} = \pm i\sqrt{y}.$$

Separating variables, integrating, using the fact that $1/i = -i$, and multiplying both sides by -1 gives

$$2i\sqrt{y} = \mp x + C_0.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\begin{aligned} &\text{Working with } -x: \quad && +C_0 = x + 2i\sqrt{y} = \phi(x,y) \\ &\text{Working with } +x: \quad && -C_0 = x - 2i\sqrt{y} = \psi(x,y). \end{aligned}$$

Because these functions are complex, however, the PDE will not be in the simplest form. Since ξ and η are complex conjugates of each other, we introduce the new real variables,

$$\begin{aligned} \alpha &= \frac{1}{2}(\xi + \eta) = x \\ \beta &= \frac{1}{2i}(\xi - \eta) = 2\sqrt{y}, \end{aligned}$$

which transform the PDE to the canonical form. After changing variables $(x, y) \to (\alpha, \beta)$, the PDE becomes

$$A^{**}u_{\alpha\alpha} + B^{**}u_{\alpha\beta} + C^{**}u_{\beta\beta} + D^{**}u_\alpha + E^{**}u_\beta + F^{**}u = G^{**},$$

where, using the chain rule,

$$\begin{aligned}
A^{**} &= A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2 \\
B^{**} &= 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y \\
C^{**} &= A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2 \\
D^{**} &= A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y \\
E^{**} &= A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y \\
F^{**} &= F \\
G^{**} &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^{**} = 1$, $B^{**} = 0$, $C^{**} = 1$, $D^{**} = 0$, $E^{**} = -\frac{1}{2\sqrt{y}} = -\frac{1}{\beta}$, $F^{**} = 0$, and $G^{**} = 0$. The PDE becomes

$$u_{\alpha\alpha} + u_{\beta\beta} - \frac{1}{\beta}u_\beta = 0.$$

Solving for $u_{\alpha\alpha} + u_{\beta\beta}$ gives

$$u_{\alpha\alpha} + u_{\beta\beta} = \frac{1}{\beta}u_\beta.$$

This is the canonical form of the elliptic PDE.

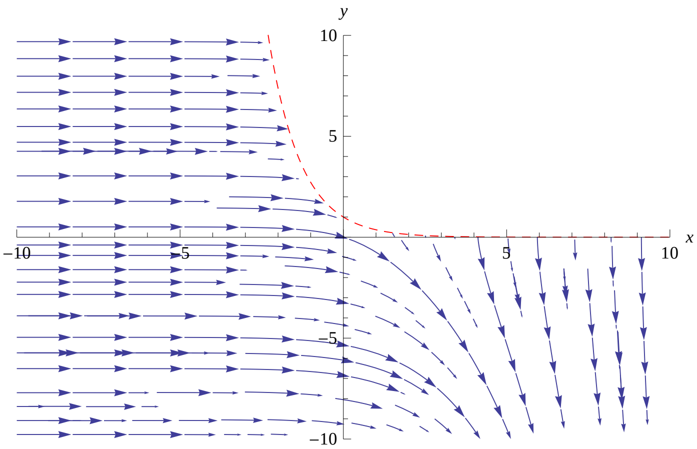

Figure 1: Slope field for $\frac{dy}{dx} = \frac{1}{y}\left(-1 + \sqrt{1 - ye^x}\right)$

Part (e)

$yu_{xx} - 2u_{xy} + e^x u_{yy} + x^2 u_x - u = 0$

Comparing this equation with the general form of a second-order PDE,
$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = y$, $B = -2$, $C = e^x$, $D = x^2$, $E = 0$, $F = -1$, and $G = 0$. The characteristic equations of this PDE are given by

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right) \\
\frac{dy}{dx} &= \frac{1}{2y}\left(-2 \pm \sqrt{4 - 4ye^x}\right) \\
\frac{dy}{dx} &= \frac{1}{y}\left(-1 \pm \sqrt{1 - ye^x}\right).
\end{aligned}$$

Note that the discriminant, $B^2 - 4AC = 4 - 4ye^x$, can be positive, zero, or negative, depending on whether $y < e^{-x}$, $y = e^{-x}$, or $y > e^{-x}$, respectively. That is,

$$\text{The PDE is } \begin{cases} \text{hyperbolic} & \text{if } y < e^{-x}. \\ \text{parabolic} & \text{if } y = e^{-x}. \\ \text{elliptic} & \text{if } y > e^{-x}. \end{cases}$$

Let us consider each case individually.

Case I: The PDE is hyperbolic ($y < e^{-x}$)

The solutions to the ordinary differential equations are two real and distinct families of characteristic curves in the xy-plane. Unfortunately, the equations are difficult (if not impossible) to solve analytically, so the canonical form of the PDE cannot be determined. The characteristic curves can be visualized, however, by plotting the slope fields for each ODE; they are tangent to the slope fields at each point on the graph. See the figures on the following page. $y(x) = e^{-x}$ is plotted in red on each graph to show the boundary of the domain of hyperbolicity.

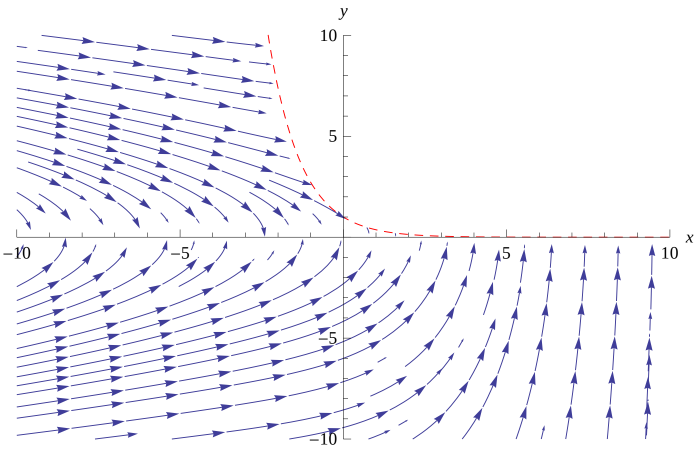

Figure 2: Slope field for $\frac{dy}{dx} = \frac{1}{y}\left(-1 - \sqrt{1 - ye^x}\right)$

Case II: The PDE is parabolic ($y = e^{-x}$)

When $y = e^{-x}$, the characteristic equations reduce to

$$\frac{dy}{dx} = -\frac{1}{y} = -e^x,$$

and this equation can be solved. Separating variables and integrating gives

$$y = -e^x + C_0.$$

Solving for the constant of integration,

$$C_0 = y + e^x = \phi(x, y).$$

Now we make the change of variables, $\xi = \phi(x, y) = y + e^x$. η can be chosen arbitrarily so long as the Jacobian of ξ and η is nonzero. We choose $\eta = y$ for simplicity. With these new variables the PDE becomes

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*,$$

where, using the chain rule, (see page 11 of the textbook for details)

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
F^* &= F \\
G^* &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = 0$, $C^* = e^x = \xi - \eta$, $D^* = e^x(x^2 + e^{-x}) = x^2 e^x + 1 = [\ln(\xi - \eta)]^2(\xi - \eta) + 1$, $E^* = 0$, $F^* = -1$, and $G^* = 0$. Thus, the PDE simplifies to

$$(\xi-\eta)u_{\eta\eta}+\left\{(\xi-\eta)[\ln(\xi-\eta)]^2+1\right\}u_\xi-u=0.$$

Solving for $u_{\eta\eta}$ gives

$$u_{\eta\eta}=-\left\{[\ln(\xi-\eta)]^2+\frac{1}{\xi-\eta}\right\}u_\xi+\frac{1}{\xi-\eta}u.$$

This is the canonical form of the parabolic PDE.

Case III: The PDE is elliptic $(y>e^{-x})$

When $y>e^{-x}$, the characteristic equations satisfy

$$\frac{dy}{dx}=\frac{1}{y}\left(-1\pm i\sqrt{ye^x-1}\right),$$

and the two distinct families of characteristic curves lie in the complex plane. If we could solve the equations, we could determine the canonical form of the PDE.

Part (f)

$u_{xx} + xu_{yy} = 0$

Comparing this equation with the general form of a second-order PDE, $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 1$, $B = 0$, $C = x$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)\\ \frac{dy}{dx} &= \frac{1}{2}\left(\pm\sqrt{-4x}\right)\\ \frac{dy}{dx} &= \pm\sqrt{-x}.\end{aligned}$$

Note that the discriminant, $B^2 - 4AC = -4x$, can be positive, zero, or negative, depending on whether $x < 0$, $x = 0$, or $x > 0$, respectively. That is,

$$\text{The PDE is} \begin{cases}\text{hyperbolic} & \text{if } x < 0.\\ \text{parabolic} & \text{if } x = 0.\\ \text{elliptic} & \text{if } x > 0.\end{cases}$$

Let us consider each case individually.

Case I: The PDE is hyperbolic $(x < 0)$

The solutions to these ordinary differential equations are two families of real and distinct characteristic curves in the xy-plane. Integrating the equations, we find that

$$y(x) = \pm\frac{2}{3}(-x)^{3/2} + C_0.$$

Solving for the constant of integration (or any convenient multiple thereof)[1],

$$\begin{aligned}&\text{Working with } -\frac{2}{3}(-x)^{3/2}: \quad C_0 = y + \frac{2}{3}(-x)^{3/2} = \phi(x, y)\\ &\text{Working with } +\frac{2}{3}(-x)^{3/2}: \quad C_0 = y - \frac{2}{3}(-x)^{3/2} = \psi(x, y).\end{aligned}$$

Now we make the change of variables, $\xi = \phi(x, y) = y + \frac{2}{3}(-x)^{3/2}$ and $\eta = \psi(x, y) = y - \frac{2}{3}(-x)^{3/2}$, so that the PDE takes the simplest form. Solving these two equations for $(-x)^{3/2}$ and y gives $(-x)^{3/2} = \frac{3}{4}(\xi - \eta)$ and $y = \frac{1}{2}(\xi + \eta)$. With these new variables the PDE becomes

$$A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} + D^*u_\xi + E^*u_\eta + F^*u = G^*,$$

[1] The book chooses to use $\frac{3}{2}C_0$ in the back, but this does not change the canonical form of the PDE.

where, using the chain rule, (see page 11 of the textbook for details)

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
F^* &= F \\
G^* &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = 4x$, $C^* = 0$, $D^* = \frac{1}{2\sqrt{-x}}$, $E^* = -\frac{1}{2\sqrt{-x}}$, $F^* = 0$, and $G^* = 0$. Thus, the PDE simplifies to

$$\begin{aligned}
4xu_{\xi\eta} + \frac{1}{2\sqrt{-x}}u_\xi - \frac{1}{2\sqrt{-x}}u_\eta &= 0 \\
u_{\xi\eta} &= -\frac{1}{4x}\left(\frac{1}{2\sqrt{-x}}u_\xi - \frac{1}{2\sqrt{-x}}u_\eta\right) \\
u_{\xi\eta} &= \frac{1}{8(-x)^{3/2}}(u_\xi - u_\eta)
\end{aligned}$$

$$u_{\xi\eta} = \frac{1}{6(\xi - \eta)}(u_\xi - u_\eta).$$

This is the first canonical form of the hyperbolic PDE. If we make the additional change of variables, $\alpha = \xi + \eta$ and $\beta = \xi - \eta$, then the chain rule gives $u_{\xi\eta} = u_{\alpha\alpha} - u_{\beta\beta}$, $u_\xi = u_\alpha + u_\beta$, and $u_\eta = u_\alpha - u_\beta$. The PDE then becomes

$$u_{\alpha\alpha} - u_{\beta\beta} = \frac{1}{6\beta}[(u_\alpha + u_\beta) - (u_\alpha - u_\beta)]$$

$$u_{\alpha\alpha} - u_{\beta\beta} = \frac{1}{3\beta}u_\beta.$$

This is the second canonical form of the hyperbolic PDE.

Case II: The PDE is parabolic $(x = 0)$

Substituting $x = 0$ into the PDE reduces it immediately to the canonical form of a parabolic equation, $u_{xx} = 0$. The characteristic equation is given by

$$\frac{dy}{dx} = 0.$$

Solving this equation for y gives $y(x) = D$, where D is an arbitrary constant. The characteristic curves in the xy-plane are lines parallel to the x-axis.

Case III: The PDE is elliptic $(x > 0)$

The characteristic equations have no real solutions for $x > 0$. This means that the two distinct families of characteristic curves lie in the complex plane. Integrating the characteristic equations, we find that

$$\frac{dy}{dx} = \pm i\sqrt{x}$$

$$y(x) = \pm\frac{2i}{3}x^{3/2} + C_0.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\text{Working with } -\frac{2i}{3}x^{3/2}: \quad +C_0 = y + \frac{2i}{3}x^{3/2} = \phi(x,y)$$

$$\text{Working with } +\frac{2i}{3}x^{3/2}: \quad +C_0 = y - \frac{2i}{3}x^{3/2} = \psi(x,y).$$

The typical variables, $\xi = \phi(x,y) = y + \frac{2i}{3}x^{3/2}$ and $\eta = \psi(x,y) = y - \frac{2i}{3}x^{3/2}$, are complex numbers, so the PDE will not transform to the simplest form. Rather, since ξ and η are complex conjugates of each other, we introduce the new real variables,

$$\alpha = \frac{1}{2}(\xi + \eta) = y$$

$$\beta = \frac{1}{2i}(\xi - \eta) = \frac{2}{3}x^{3/2},$$

which do transform the PDE to the simplest form. After changing variables $(x,y) \to (\alpha,\beta)$, the PDE becomes

$$A^{**}u_{\alpha\alpha} + B^{**}u_{\alpha\beta} + C^{**}u_{\beta\beta} + D^{**}u_\alpha + E^{**}u_\beta + F^{**}u = G^{**},$$

where, using the chain rule,

$$\begin{aligned}
A^{**} &= A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2 \\
B^{**} &= 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y \\
C^{**} &= A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2 \\
D^{**} &= A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y \\
E^{**} &= A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y \\
F^{**} &= F \\
G^{**} &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these equations, we find that $A^{**} = x$, $B^{**} = 0$, $C^{**} = x$, $D^{**} = 0$, $E^{**} = \frac{1}{2\sqrt{x}}$, $F^{**} = 0$, and $G^{**} = 0$. The PDE becomes

$$xu_{\alpha\alpha} + xu_{\beta\beta} + \frac{1}{2\sqrt{x}}u_\beta = 0$$

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{2x^{3/2}}u_\beta$$

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{3\beta}u_\beta.$$

This is the canonical form of the elliptic PDE.

Part (g)

$x^2u_{xx} + 4yu_{xy} + u_{yy} + 2u_x = 0$

Comparing this equation with the general form of a second-order PDE,
$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = x^2$, $B = 4y$, $C = 1$, $D = 2$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)\\ \frac{dy}{dx} &= \frac{1}{2x^2}\left(4y \pm \sqrt{16y^2 - 4x^2}\right)\\ \frac{dy}{dx} &= \frac{1}{x^2}\left(2y \pm \sqrt{4y^2 - x^2}\right).\end{aligned}$$

Note that the discriminant, $B^2 - 4AC = 16y^2 - 4x^2$, can be positive, zero, or negative, depending on whether $16y^2 - 4x^2 > 0$, $16y^2 - 4x^2 = 0$, or $16y^2 - 4x^2 < 0$, respectively. That is,[2]

$$\text{The PDE is } \begin{cases} \text{hyperbolic} & \text{if } 2|y| > |x|. \\ \text{parabolic} & \text{if } 2|y| = |x|. \\ \text{elliptic} & \text{if } 2|y| < |x|. \end{cases}$$

Let us consider each case individually.

Case I: The PDE is hyperbolic $2|y| > |x|$

The solutions to the ordinary differential equations are two real and distinct families of characteristic curves in the xy-plane. Unfortunately, the equations are difficult (if not impossible) to solve analytically, so the canonical form of the PDE cannot be determined. The characteristic curves can be visualized, however, by plotting the slope fields for each ODE; they are tangent to the slope fields at each point on the graph. See the figures on the following page. $2|y| = |x|$ is plotted in red on each graph to show the boundary of the domain of hyperbolicity.

[2] Bring $4x^2$ to the right, take the square root of both sides, and divide both sides by 2 in all cases.

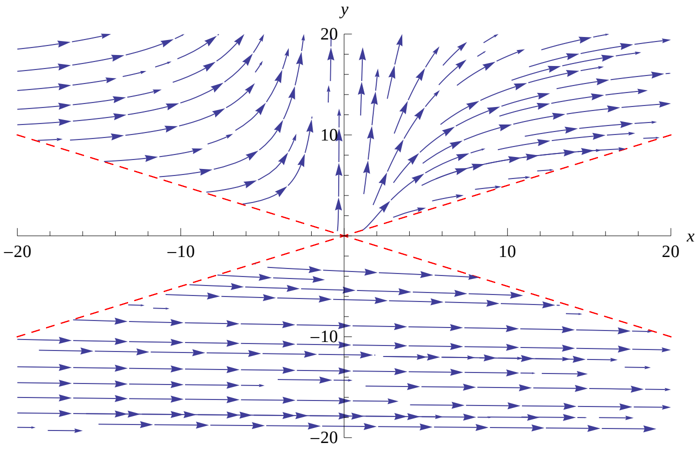

Figure 3: Slope field for $\frac{dy}{dx} = \frac{1}{x^2}\left(2y + \sqrt{4y^2 - x^2}\right)$

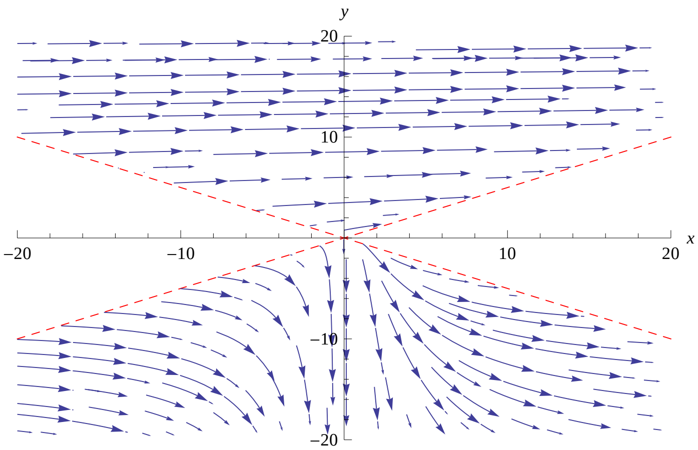

Figure 4: Slope field for $\frac{dy}{dx} = \frac{1}{x^2}\left(2y - \sqrt{4y^2 - x^2}\right)$

Case II: The PDE is parabolic $(2|y| = |x|)$

Squaring both sides gives $4y^2 = x^2$, and the characteristic equations,

$$\frac{dy}{dx} = \frac{1}{x^2}\left(2y \pm \sqrt{4y^2 - x^2}\right),$$

reduce to

$$\frac{dy}{dx} = \frac{2y}{x^2} = \frac{1}{2y}.$$

and this equation can be solved. Separating variables and integrating gives

$$y^2 = x + C_0.$$

Solving for the constant of integration,

$$C_0 = y^2 - x = \phi(x, y).$$

Now we make the change of variables, $\xi = \phi(x, y) = y^2 - x$. η can be chosen arbitrarily so long as the Jacobian of ξ and η is nonzero. We choose $\eta = y$ for simplicity. With these new variables the PDE becomes

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*,$$

where, using the chain rule, (see page 11 of the textbook for details)

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
F^* &= F \\
G^* &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = 0$, $C^* = 1$, $D^* = 0$, $E^* = 0$, $F^* = 0$, and $G^* = 0$. Thus, the PDE simplifies to

$$u_{\eta\eta} = 0.$$

This is the canonical form of the parabolic PDE.

Case III: The PDE is elliptic $(2|y| < |x|)$

When $4y^2 - x^2 < 0$, the characteristic equations satisfy

$$\frac{dy}{dx} = \frac{1}{x^2}\left(2y \pm i\sqrt{x^2 - 4y^2}\right),$$

and the two distinct families of characteristic curves lie in the complex plane. If we could solve the equations, we could determine the canonical form of the elliptic PDE.

Part (h)

$3yu_{xx} - xu_{yy} = 0$

Comparing this equation with the general form of a second-order PDE,
$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 3y$, $B = 0$, $C = -x$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\frac{dy}{dx} = \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)$$
$$\frac{dy}{dx} = \frac{1}{6y}\left(\pm\sqrt{12xy}\right)$$
$$\frac{dy}{dx} = \pm\sqrt{\frac{x}{3y}}.$$

Note that the discriminant, $B^2 - 4AC = 12xy$, can be positive, zero, or negative, depending on whether $xy > 0$, $xy = 0$, or $xy < 0$, respectively. That is,

$$\text{The PDE is } \begin{cases} \text{hyperbolic} & \text{if } xy > 0. \\ \text{parabolic} & \text{if } xy = 0. \\ \text{elliptic} & \text{if } xy < 0. \end{cases}$$

Let us consider each case individually.

Case I: The PDE is hyperbolic $(xy > 0)$[3]

The solutions to these ordinary differential equations are two real and distinct families of characteristic curves in the xy-plane. Separating variables and integrating the equations, we find that

$$\frac{2}{3}y^{3/2} = \pm\frac{1}{\sqrt{3}} \cdot \frac{2}{3}x^{3/2} + C_0.$$

The characteristic curves are given by

$$y(x) = \left(\frac{3}{2}C_0 \pm \frac{1}{\sqrt{3}}x^{3/2}\right)^{2/3}.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\text{Working with } -\frac{1}{\sqrt{3}} \cdot \frac{2}{3}x^{3/2}: \quad \frac{3}{2}C_0 = y^{3/2} + \frac{1}{\sqrt{3}}x^{3/2} = \phi(x, y)$$
$$\text{Working with } +\frac{1}{\sqrt{3}} \cdot \frac{2}{3}x^{3/2}: \quad \frac{3}{2}C_0 = y^{3/2} - \frac{1}{\sqrt{3}}x^{3/2} = \psi(x, y).$$

Now we make the change of variables, $\xi = \phi(x, y) = y^{3/2} + \frac{1}{\sqrt{3}}x^{3/2}$ and $\eta = \psi(x, y) = y^{3/2} - \frac{1}{\sqrt{3}}x^{3/2}$, so that the PDE takes the simplest form. Solving these two equations for x and y gives $x = \left[\frac{\sqrt{3}}{2}(\xi - \eta)\right]^{2/3}$ and $y = \left[\frac{1}{2}(\xi + \eta)\right]^{2/3}$. With these new variables the PDE becomes

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*,$$

[3] $xy > 0$ holds in the first and third quadrants.

where, using the chain rule, (see page 11 of the textbook for details)

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
F^* &= F \\
G^* &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $C^* = 0$, $F^* = 0$, $G^* = 0$,

$$\begin{aligned}
B^* &= -9xy = -\frac{9}{2}\left(\frac{3}{2}\right)^{1/3}(\xi-\eta)^{2/3}(\xi+\eta)^{2/3}, \\
D^* &= \frac{3\sqrt{3}}{4}\frac{y}{\sqrt{x}} - \frac{3x}{4\sqrt{y}} = \frac{3\left(\frac{3}{2}\right)^{1/3}\eta}{2(\xi-\eta)^{1/3}(\xi+\eta)^{1/3}}, \\
E^* &= -\frac{3\sqrt{3}}{4}\frac{y}{\sqrt{x}} - \frac{3x}{4\sqrt{y}} = -\frac{3\left(\frac{3}{2}\right)^{1/3}\xi}{2(\xi-\eta)^{1/3}(\xi+\eta)^{1/3}}.
\end{aligned}$$

Thus, the PDE simplifies to

$$-\frac{9}{2}\left(\frac{3}{2}\right)^{1/3}(\xi-\eta)^{2/3}(\xi+\eta)^{2/3}u_{\xi\eta} + \frac{3\left(\frac{3}{2}\right)^{1/3}\eta}{2(\xi-\eta)^{1/3}(\xi+\eta)^{1/3}}u_\xi - \frac{3\left(\frac{3}{2}\right)^{1/3}\xi}{2(\xi-\eta)^{1/3}(\xi+\eta)^{1/3}}u_\eta = 0.$$

Solving for $u_{\xi\eta}$ gives

$$u_{\xi\eta} = \frac{\xi}{3(\xi^2-\eta^2)}(\eta u_\xi - \xi u_\eta).$$

This is the first canonical form of the hyperbolic PDE. If we make the additional change of variables, $\alpha = \xi + \eta$ and $\beta = \xi - \eta$, then the chain rule gives $u_{\xi\eta} = u_{\alpha\alpha} - u_{\beta\beta}$, $u_\xi = u_\alpha + u_\beta$, and $u_\eta = u_\alpha - u_\beta$. Solving these two equations for ξ and η gives $\xi = (\alpha+\beta)/2$ and $\eta = (\alpha-\beta)/2$. The PDE then becomes

$$u_{\alpha\alpha} - u_{\beta\beta} = \frac{\alpha+\beta}{6\alpha\beta}(\alpha u_\beta - \beta u_\alpha).$$

This is the second canonical form of the hyperbolic PDE.

Case II: The PDE is parabolic $(xy = 0)$[4]

Substituting $x = 0$ or $y = 0$ into the PDE reduces it immediately to the canonical form of a parabolic equation, $u_{\eta\eta} = 0$. The characteristic equation reduces to

$$\frac{dy}{dx} = 0.$$

Solving this equation for y gives $y(x) = D$, where D is an arbitrary constant. The characteristic curves in the xy-plane are lines parallel to the x-axis.

Case III: The PDE is elliptic $(xy < 0)$[5]

The characteristic equations have no real solutions for $xy < 0$. This means that the two distinct families of characteristic curves lie in the complex plane. Separating variables and integrating the characteristic equations, we find that

$$\frac{dy}{dx} = \pm i\sqrt{\frac{x}{3y}}$$
$$\frac{2}{3}y^{3/2} = \pm\frac{i}{\sqrt{3}} \cdot \frac{2}{3}x^{3/2} + C_0.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\text{Working with } -\frac{i}{\sqrt{3}} \cdot \frac{2}{3}x^{3/2}: \quad \frac{3}{2}C_0 = y^{3/2} + \frac{i}{\sqrt{3}}x^{3/2} = \phi(x, y)$$
$$\text{Working with } +\frac{i}{\sqrt{3}} \cdot \frac{2}{3}x^{3/2}: \quad \frac{3}{2}C_0 = y^{3/2} - \frac{i}{\sqrt{3}}x^{3/2} = \psi(x, y).$$

The variables, $\xi = \phi(x, y) = y^{3/2} + \frac{i}{\sqrt{3}}x^{3/2}$ and $\eta = \psi(x, y) = y^{3/2} - \frac{i}{\sqrt{3}}x^{3/2}$, are complex conjugates of one another, so we introduce the new real variables $\alpha = (\xi + \eta)/2$ and $\beta = (\xi - \eta)/2i$. They are[6]

$$\alpha = \frac{1}{2}(\xi + \eta) = y^{3/2},$$
$$\beta = \frac{1}{2i}(\xi - \eta) = \frac{1}{\sqrt{3}}(-x)^{3/2},$$

After changing variables $(x, y) \to (\alpha, \beta)$, the PDE becomes

$$A^{**}u_{\alpha\alpha} + B^{**}u_{\alpha\beta} + C^{**}u_{\beta\beta} + D^{**}u_\alpha + E^{**}u_\beta + F^{**}u = G^{**},$$

[4] $xy = 0$ holds on the x and y axes.

[5] $xy < 0$ holds in the second and fourth quadrants.

[6] Because $xy < 0$, one and only one of the variables needs to have a negative sign when changing variables. Choose $+y$ and $-x$ for this exercise. If this is not done, the canonical form of the elliptic PDE will not be obtained.

where, using the chain rule,

$$\begin{aligned}
A^{**} &= A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2 \\
B^{**} &= 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y \\
C^{**} &= A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2 \\
D^{**} &= A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y \\
E^{**} &= A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y \\
F^{**} &= F \\
G^{**} &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas gives $A^{**} = -\frac{9xy}{4}$, $B^{**} = 0$, $C^{**} = -\frac{9xy}{4}$, $D^{**} = -\frac{3x}{4\sqrt{y}}$, $E^{**} = \frac{3\sqrt{3}y}{4\sqrt{-x}}$, $F^{**} = 0$, and $G^{**} = 0$. The PDE simplifies to

$$-\frac{9xy}{4}u_{\alpha\alpha} - \frac{9xy}{4}u_{\beta\beta} - \frac{3x}{4\sqrt{y}}u_\alpha + \frac{3\sqrt{3}y}{4\sqrt{-x}}u_\beta = 0$$

$$u_{\alpha\alpha} + u_{\beta\beta} + \frac{1}{3y^{3/2}}u_\alpha + \frac{1}{\sqrt{3}(-x)^{3/2}}u_\beta = 0$$

$$u_{\alpha\alpha} + u_{\beta\beta} + \frac{1}{3\alpha}u_\alpha + \frac{1}{3\beta}u_\beta = 0.$$

Solving for $u_{\alpha\alpha} + u_{\beta\beta}$ gives

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{3}\left(\frac{u_\alpha}{\alpha} + \frac{u_\beta}{\beta}\right).$$

This is the canonical form of the elliptic PDE.

Part (i)

$u_{xx} + 2xu_{xy} + a^2u_{yy} + u = 5$

Comparing this equation with the general form of a second-order PDE,
$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 1$, $B = 2x$, $C = a^2$, $D = 0$, $E = 0$, $F = 1$, and $G = 5$. The characteristic equations of this PDE are given by

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right) \\
\frac{dy}{dx} &= \frac{1}{2}\left(2x \pm \sqrt{4x^2 - 4a^2}\right) \\
\frac{dy}{dx} &= x \pm \sqrt{x^2 - a^2}.
\end{aligned}$$

Note that the discriminant, $B^2 - 4AC = 4x^2 - 4a^2$, can be positive, zero, or negative, depending on whether $x^2 - a^2 > 0$, $x^2 - a^2 = 0$, or $x^2 - a^2 < 0$, respectively. That is,[7]

$$\text{The PDE is } \begin{cases} \text{hyperbolic} & \text{if } |x| > |a|. \\ \text{parabolic} & \text{if } |x| = |a|. \\ \text{elliptic} & \text{if } |x| < |a|. \end{cases}$$

Let us consider each case individually.

Case I: The PDE is hyperbolic ($|x| > |a|$)

The solutions to the ordinary differential equations are two real and distinct families of characteristic curves in the xy-plane. Integrating the characteristic equations, we find that

$$y(x) = \frac{1}{2}\left[x\left(x \pm \sqrt{x^2 - a^2}\right) \mp a^2 \ln\left(x + \sqrt{x^2 - a^2}\right)\right] + C_0.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\begin{aligned}
&\text{Working with } - \text{ and } +: \quad 2C_0 = 2y - \left[x\left(x - \sqrt{x^2 - a^2}\right) + a^2 \ln\left(x + \sqrt{x^2 - a^2}\right)\right] = \phi(x, y) \\
&\text{Working with } + \text{ and } -: \quad 2C_0 = 2y - \left[x\left(x + \sqrt{x^2 - a^2}\right) - a^2 \ln\left(x + \sqrt{x^2 - a^2}\right)\right] = \psi(x, y).
\end{aligned}$$

Now we make the change of variables,
$\xi = \phi(x, y) = 2y - \left[x\left(x - \sqrt{x^2 - a^2}\right) + a^2 \ln\left(x + \sqrt{x^2 - a^2}\right)\right]$ and
$\eta = \psi(x, y) = 2y - \left[x\left(x + \sqrt{x^2 - a^2}\right) - a^2 \ln\left(x + \sqrt{x^2 - a^2}\right)\right]$, so that the PDE takes the simplest form. By eliminating y and solving for x, we obtain the transcendental equation,
$\xi - \eta = 2x\sqrt{x^2 - a^2} - 2a^2 \ln\left(x + \sqrt{x^2 - a^2}\right)$. With the change of variables $(x, y) \to (\xi, \eta)$, the PDE becomes

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*,$$

[7] Bring a^2 to the right and take the square root of both sides.

where, using the chain rule, (see page 11 of the textbook for details)

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
F^* &= F \\
G^* &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = 16\left(a^2 - x^2\right)$, $C^* = 0$, $D^* = -2 + \frac{2x}{\sqrt{x^2-a^2}}$, $E^* = -2 - \frac{2x}{\sqrt{x^2-a^2}}$, $F^* = 1$, and $G^* = 5$. Thus, the PDE simplifies to

$$16\left(a^2 - x^2\right)u_{\xi\eta} + \left(-2 + \frac{2x}{\sqrt{x^2-a^2}}\right)u_\xi + \left(-2 - \frac{2x}{\sqrt{x^2-a^2}}\right)u_\eta + u = 5$$

$$u_{\xi\eta} = \frac{1}{16(x^2-a^2)}\left[2\left(\frac{x}{\sqrt{x^2-a^2}} - 1\right)u_\xi - 2\left(\frac{x}{\sqrt{x^2-a^2}} + 1\right)u_\eta + u - 5\right].$$

This is the first canonical form of the hyperbolic PDE. Since the transcendental equation cannot be solved for x explicitly, we leave the PDE in terms of x. If we make the additional change of variables, $\alpha = \xi + \eta$ and $\beta = \xi - \eta$, then the chain rule gives $u_{\xi\eta} = u_{\alpha\alpha} - u_{\beta\beta}$, $u_\xi = u_\alpha + u_\beta$, and $u_\eta = u_\alpha - u_\beta$. The PDE then becomes

$$u_{\alpha\alpha} - u_{\beta\beta} = \frac{1}{16(x^2-a^2)}\left[2\left(\frac{x}{\sqrt{x^2-a^2}} - 1\right)(u_\alpha + u_\beta) - 2\left(\frac{x}{\sqrt{x^2-a^2}} + 1\right)(u_\alpha - u_\beta) + u - 5\right]$$

$$u_{\alpha\alpha} - u_{\beta\beta} = \frac{1}{16(x^2-a^2)}\left(-4u_\alpha + \frac{4x}{x^2-a^2}u_\beta + u - 5\right),$$

where $\beta = 2x\sqrt{x^2-a^2} - 2a^2\ln\left(x + \sqrt{x^2-a^2}\right)$. This is the second canonical form of the hyperbolic PDE.

Case II: The PDE is parabolic $(|x| = |a|)$

When $x^2 - a^2 = 0$, the equations for the characteristics reduce to

$$\frac{dy}{dx} = x.$$

Integrating this gives the characteristic curves:

$$y(x) = \frac{1}{2}x^2 + C_0.$$

Solving now for the integration constant,

$$C_0 = y - \frac{1}{2}x^2 = \phi(x,y).$$

Now we make the change of variables, $\xi = \phi(x, y) = y - \frac{1}{2}x^2$. η can be chosen arbitrarily so long as the Jacobian of ξ and η is nonzero. We choose $\eta = y$ for simplicity. Solving these two equations for x and y gives $x^2 = 2(\eta - \xi)$ and $y = \eta$. With these new variables the PDE becomes

$$A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} + D^*u_\xi + E^*u_\eta + F^*u = G^*,$$

where, using the chain rule, (see page 11 of the textbook for details)

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
F^* &= F \\
G^* &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = a^2 - x^2 = 0$, $B^* = 2(a^2 - x^2) = 0$, $C^* = a^2$, $D^* = -1$, $E^* = 0$, $F^* = 1$, and $G^* = 5$. Thus, the PDE simplifies to

$$\begin{aligned}
a^2u_{\eta\eta} - u_\xi + u &= 5 \\
u_{\eta\eta} &= \frac{1}{a^2}(u_\xi - u + 5).
\end{aligned}$$

This is the canonical form of the parabolic PDE.

Case III: The PDE is elliptic $(x^2 - a^2 < 0)$

The characteristic equations have no real solutions in this case. This means that the two distinct families of characteristic curves lie in the complex plane. Integrating the characteristic equations, we find that

$$\begin{aligned}
\frac{dy}{dx} &= x \pm i\sqrt{a^2 - x^2} \\
y(x) &= \frac{1}{2}\left[x^2 \pm i\left(x\sqrt{a^2 - x^2} + a^2\tan^{-1}\frac{x}{\sqrt{a^2 - x^2}}\right)\right] + C_0.
\end{aligned}$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\begin{aligned}
&\text{Working with } -i\text{:} \quad 2C_0 = 2y - x^2 + i\left(x\sqrt{a^2 - x^2} + a^2\tan^{-1}\frac{x}{\sqrt{a^2 - x^2}}\right) = \phi(x, y) \\
&\text{Working with } +i\text{:} \quad 2C_0 = 2y - x^2 - i\left(x\sqrt{a^2 - x^2} + a^2\tan^{-1}\frac{x}{\sqrt{a^2 - x^2}}\right) = \psi(x, y).
\end{aligned}$$

The variables, $\xi = \phi(x, y) = 2y - x^2 + i\left(x\sqrt{a^2 - x^2} + a^2\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}\right)$ and $\eta = \psi(x, y) = 2y - x^2 - i\left(x\sqrt{a^2 - x^2} + a^2\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}\right)$, are complex conjugates of each other, so we introduce the new real variables,

$$\begin{aligned}
\alpha &= \frac{1}{2}(\xi + \eta) = 2y - x^2 \\
\beta &= \frac{1}{2i}(\xi - \eta) = x\sqrt{a^2 - x^2} + a^2\tan^{-1}\frac{x}{\sqrt{a^2 - x^2}},
\end{aligned}$$

which transform the PDE to

$$A^{**}u_{\alpha\alpha} + B^{**}u_{\alpha\beta} + C^{**}u_{\beta\beta} + D^{**}u_{\alpha} + E^{**}u_{\beta} + F^{**}u = G^{**},$$

where, using the chain rule,

$$\begin{aligned}
A^{**} &= A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2 \\
B^{**} &= 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y \\
C^{**} &= A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2 \\
D^{**} &= A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y \\
E^{**} &= A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y \\
F^{**} &= F \\
G^{**} &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas gives $A^{**} = 4\left(a^2 - x^2\right)$, $B^{**} = 0$, $C^{**} = 4\left(a^2 - x^2\right)$, $D^{**} = -2$, $E^{**} = -\frac{2x}{\sqrt{a^2-x^2}}$, $F^{**} = 1$, and $G^{**} = 5$. The PDE becomes

$$4\left(a^2 - x^2\right)u_{\alpha\alpha} + 4\left(a^2 - x^2\right)u_{\beta\beta} - 2u_{\alpha} - \frac{2x}{\sqrt{a^2 - x^2}}u_{\beta} + u = 5$$

$$4\left(a^2 - x^2\right)(u_{\alpha\alpha} + u_{\beta\beta}) = 2u_{\alpha} + \frac{2x}{\sqrt{a^2 - x^2}}u_{\beta} - u + 5$$

$$u_{\alpha\alpha} + u_{\beta\beta} = \frac{1}{4\left(a^2 - x^2\right)}\left(2u_{\alpha} + \frac{2x}{\sqrt{a^2 - x^2}}u_{\beta} - u + 5\right).$$

This is the canonical form of the elliptic PDE, where x is defined implicitly in terms of β.

Part (j)

$y^2 u_{xx} + x^2 u_{yy} = 0$

Comparing this equation with the general form of a second-order PDE,
$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = y^2$, $B = 0$, $C = x^2$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)\\ \frac{dy}{dx} &= \frac{1}{2y^2}\left(\pm\sqrt{-4x^2y^2}\right)\\ \frac{dy}{dx} &= \pm\frac{ix}{y}.\end{aligned}$$

Note that $B^2 - 4AC = -4x^2y^2 < 0$, which means that the PDE is **elliptic** for all x and y. Therefore, the solutions to the ordinary differential equations are two distinct families of characteristic curves that lie in the complex plane. Separating variables and integrating the equations, we find that

$$\frac{1}{2}y^2 = \pm\frac{i}{2}x^2 + C_0.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\begin{aligned}&\text{Working with } -\frac{i}{2}x^2: \quad 2C_0 = y^2 + ix^2 = \phi(x, y)\\ &\text{Working with } +\frac{i}{2}x^2: \quad 2C_0 = y^2 - ix^2 = \psi(x, y).\end{aligned}$$

The typical variables, $\xi = \phi(x, y) = y^2 + ix^2$ and $\eta = \psi(x, y) = y^2 - ix^2$, are complex numbers, so the PDE will not transform to the simplest form. Rather, since ξ and η are complex conjugates of each other, we introduce the new real variables,

$$\begin{aligned}\alpha &= \frac{1}{2}(\xi + \eta) = y^2\\ \beta &= \frac{1}{2i}(\xi - \eta) = x^2,\end{aligned}$$

which do transform the PDE to the simplest form. After changing variables $(x, y) \to (\alpha, \beta)$, the PDE becomes

$$A^{**}u_{\alpha\alpha} + B^{**}u_{\alpha\beta} + C^{**}u_{\beta\beta} + D^{**}u_\alpha + E^{**}u_\beta + F^{**}u = G^{**},$$

where, using the chain rule,

$$\begin{aligned}A^{**} &= A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2\\ B^{**} &= 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y\\ C^{**} &= A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2\\ D^{**} &= A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y\\ E^{**} &= A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y\\ F^{**} &= F\\ G^{**} &= G.\end{aligned}$$

Plugging in the numbers and derivatives to these equations, $A^{**} = 4x^2y^2 = 4\alpha\beta$, $B^{**} = 0$, $C^{**} = 4x^2y^2 = 4\alpha\beta$, $D^{**} = 2x^2 = 2\beta$, $E^{**} = 2y^2 = 2\alpha$, $F^{**} = 0$, and $G^{**} = 0$. So the PDE becomes

$$\begin{gathered}
4\alpha\beta u_{\alpha\alpha} + 4\alpha\beta u_{\beta\beta} + 2\beta u_\alpha + 2\alpha u_\beta = 0 \\
u_{\alpha\alpha} + u_{\beta\beta} + \frac{1}{2\alpha} u_\alpha + \frac{1}{2\beta} u_\beta = 0 \\
u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{2}\left(\frac{u_\alpha}{\alpha} + \frac{u_\beta}{\beta}\right).
\end{gathered}$$

This is the canonical form of the elliptic PDE.