Exercise 2

Determine the nature of the following equations and reduce them to canonical form:

(a) $x^2u_{xx} + 4xyu_{xy} + y^2u_{yy} = 0$

(b) $u_{xx} - xu_{yy} = 0$

(c) $u_{xx} - 2u_{xy} + 3u_{yy} + 24u_y + 5u = 0$

(d) $u_{xx} + \operatorname{sech}^4 x u_{yy} = 0$

(e) $u_{xx} + 6yu_{xy} + 9y^2u_{yy} + 4u = 0$

(f) $u_{xx} - \operatorname{sech}^4 x u_{yy} = 0$

(g) $u_{xx} + 2\csc y u_{xy} + \csc^2 y u_{yy} = 0$

(h) $u_{xx} - 5u_{xy} + 5u_{yy} = 0$

Solution

Part (a)

$x^2u_{xx} + 4xyu_{xy} + y^2u_{yy} = 0$

Comparing this equation with the general form of a second-order PDE, $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = x^2$, $B = 4xy$, $C = y^2$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right) \\
\frac{dy}{dx} &= \frac{1}{2x^2}\left(4xy \pm \sqrt{16x^2y^2 - 4x^2y^2}\right) \\
\frac{dy}{dx} &= \frac{1}{2x^2}\left(4xy \pm 2xy\sqrt{3}\right) \\
\frac{dy}{dx} &= \frac{y}{x}\left(2 \pm \sqrt{3}\right).
\end{aligned}$$

Note that the discriminant, $B^2 - 4AC = 12x^2y^2$, is greater than 0 for all x and y, which means that the PDE is **hyperbolic**. The solutions to the ordinary differential equations are therefore two distinct families of real characteristic curves in the xy-plane. Separating variables and integrating the equations, we find that

$$\ln|y| = \left(2 \pm \sqrt{3}\right)\ln|x| + C_0.$$

Exponentiating both sides gives us the characteristic curves:

$$y(x) = A_0|x|^{\left(2\pm\sqrt{3}\right)}.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\begin{aligned}
\text{Working with } 2-\sqrt{3}:\quad C_0 &= \ln|y| - \left(2 - \sqrt{3}\right)\ln|x| = \phi(x,y) \\
\text{Working with } 2+\sqrt{3}:\quad C_0 &= \ln|y| - \left(2 + \sqrt{3}\right)\ln|x| = \psi(x,y).
\end{aligned}$$

Now we make the change of variables, $\xi = \phi(x, y) = \ln|y| - \left(2 - \sqrt{3}\right)\ln|x|$ and $\eta = \psi(x, y) = \ln|y| - \left(2 + \sqrt{3}\right)\ln|x|$, so that the PDE takes the simplest form. With these new variables the PDE becomes

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*,$$

where, using the chain rule, (see page 11 of the textbook for details)

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
F^* &= F \\
G^* &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = -12$, $C^* = 0$, $D^* = 1 - \sqrt{3}$, $E^* = 1 + \sqrt{3}$, $F^* = 0$, and $G^* = 0$. Thus, the PDE simplifies to

$$-12u_{\xi\eta} + \left(1 - \sqrt{3}\right)u_\xi + \left(1 + \sqrt{3}\right)u_\eta = 0.$$

Solving for $u_{\xi\eta}$ gives

$$u_{\xi\eta} = \frac{1}{12}\left[\left(1 - \sqrt{3}\right)u_\xi + \left(1 + \sqrt{3}\right)u_\eta\right].$$

This is the first canonical form of the hyperbolic PDE. To get the second canonical form, we have to make the additional change of variables, $\alpha = \xi + \eta$ and $\beta = \xi - \eta$. The chain rule gives $u_{\xi\eta} = u_{\alpha\alpha} - u_{\beta\beta}$, $u_\xi = u_\alpha + u_\beta$, and $u_\eta = u_\alpha - u_\beta$. The PDE becomes

$$u_{\alpha\alpha} - u_{\beta\beta} = \frac{1}{6}(u_\alpha - \sqrt{3}u_\beta).$$

This is the second canonical form of the hyperbolic PDE.

Part (b)

$u_{xx} - xu_{yy} = 0$

Comparing this equation with the general form of a second-order PDE, $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 1$, $B = 0$, $C = -x$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)\\ \frac{dy}{dx} &= \frac{1}{2}\left(\pm\sqrt{4x}\right)\\ \frac{dy}{dx} &= \pm\sqrt{x}.\end{aligned}$$

Note that the discriminant, $B^2 - 4AC = 4x$, can be positive, zero, or negative, depending on whether $x > 0$, $x = 0$, or $x < 0$, respectively. That is,

$$\text{The PDE is} \begin{cases} \text{hyperbolic} & \text{if } x > 0. \\ \text{parabolic} & \text{if } x = 0. \\ \text{elliptic} & \text{if } x < 0. \end{cases}$$

Let us consider each case individually.

Case I: The PDE is hyperbolic $(x > 0)$

The solutions to these ordinary differential equations are two distinct families of real characteristic curves in the xy-plane. Integrating the equations, we find that

$$y(x) = \pm\frac{2}{3}x^{3/2} + C_0.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\begin{aligned}\text{Working with } -\frac{2}{3}x^{3/2}: \quad C_0 = y + \frac{2}{3}x^{3/2} = \phi(x, y)\\ \text{Working with } +\frac{2}{3}x^{3/2}: \quad C_0 = y - \frac{2}{3}x^{3/2} = \psi(x, y).\end{aligned}$$

Now we make the change of variables, $\xi = \phi(x, y) = y + \frac{2}{3}x^{3/2}$ and $\eta = \psi(x, y) = y - \frac{2}{3}x^{3/2}$, so that the PDE takes the simplest form. Solving these two equations for x and y gives $x^{3/2} = \frac{3}{4}(\xi - \eta)$ and $y = \frac{1}{2}(\xi + \eta)$. With these new variables the PDE becomes

$$A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} + D^*u_\xi + E^*u_\eta + F^*u = G^*,$$

where, using the chain rule, (see page 11 of the textbook for details)

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
F^* &= F \\
G^* &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = -4x$, $C^* = 0$, $D^* = \frac{1}{2\sqrt{x}}$, $E^* = -\frac{1}{2\sqrt{x}}$, $F^* = 0$, and $G^* = 0$. Thus, the PDE simplifies to

$$-4xu_{\xi\eta} + \frac{1}{2\sqrt{x}}u_\xi - \frac{1}{2\sqrt{x}}u_\eta = 0.$$

Solving now for $u_{\xi\eta}$ gives

$$u_{\xi\eta} = \frac{1}{8x^{3/2}}(u_\xi - u_\eta),$$

which is

$$u_{\xi\eta} = \frac{1}{6(\xi - \eta)}(u_\xi - u_\eta).$$

This is the first canonical form of the hyperbolic PDE. To get the second canonical form, we have to make the additional change of variables, $\alpha = \xi + \eta$ and $\beta = \xi - \eta$. The chain rule gives $u_{\xi\eta} = u_{\alpha\alpha} - u_{\beta\beta}$, $u_\xi = u_\alpha + u_\beta$, and $u_\eta = u_\alpha - u_\beta$. Changing variables gives us

$$u_{\alpha\alpha} - u_{\beta\beta} = \frac{1}{3\beta}u_\beta.$$

This is the second canonical form of the hyperbolic PDE.

Case II: The PDE is parabolic $(x = 0)$

Substituting $x = 0$ into the PDE reduces it immediately to the canonical form of a parabolic equation, $u_{xx} = 0$. The characteristic equations reduce to

$$\frac{dy}{dx} = 0.$$

The characteristic curves in the xy-plane are lines parallel to the x-axis, $y(x) = C_0$, where C_0 is an arbitrary constant.

Case III: The PDE is elliptic $(x < 0)$

The characteristic equations have no real solutions for $x < 0$. This means that the two distinct families of characteristic curves lie in the complex plane. Integrating the characteristic equations, we find that

$$\frac{dy}{dx} = \pm i\sqrt{x}$$

$$y(x) = \pm\frac{2i}{3}x^{3/2} + C_0.$$

Solving for the constant of integration, C_0 (or any convenient multiple thereof),

$$\begin{aligned} \text{Working with } -\frac{2i}{3}x^{3/2}: &\quad C_0 = y + \frac{2i}{3}x^{3/2} = \phi(x, y) \\ \text{Working with } +\frac{2i}{3}x^{3/2}: &\quad C_0 = y - \frac{2i}{3}x^{3/2} = \psi(x, y). \end{aligned}$$

Because $\xi = \phi(x, y) = y + \frac{2i}{3}x^{3/2}$ and $\eta = \psi(x, y) = y - \frac{2i}{3}x^{3/2}$ are complex conjugates of one another, we introduce the new real variables[1],

$$\begin{aligned} \alpha &= \frac{\xi + \eta}{2} = y \\ \beta &= \frac{\xi - \eta}{2i} = \frac{2}{3}(-x)^{3/2}, \end{aligned}$$

which transform the PDE to the canonical form. After changing variables $(x, y) \to (\alpha, \beta)$, the PDE becomes

$$A^{**}u_{\alpha\alpha} + B^{**}u_{\alpha\beta} + C^{**}u_{\beta\beta} + D^{**}u_\alpha + E^{**}u_\beta + F^{**}u = G^{**},$$

where, using the chain rule,

$$\begin{aligned} A^{**} &= A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2 \\ B^{**} &= 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y \\ C^{**} &= A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2 \\ D^{**} &= A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y \\ E^{**} &= A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y \\ F^{**} &= F \\ G^{**} &= G. \end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^{**} = -x$, $B^{**} = 0$, $C^{**} = -x$, $D^{**} = 0$, $E^{**} = \frac{1}{2\sqrt{-x}}$, $F^{**} = 0$, and $G^{**} = 0$. Thus, the PDE simplifies to

$$\begin{gathered} -xu_{\alpha\alpha} - xu_{\beta\beta} + \frac{1}{2\sqrt{-x}}u_\beta = 0 \\ u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{2(-x)^{3/2}}u_\beta, \end{gathered}$$

which is

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{3\beta}u_\beta.$$

This is the canonical form of the elliptic PDE.

[1] Since $x < 0$, we have to use $-x$ in the change of variables. Otherwise, we will not get the desired canonical form.

Part (c)

$u_{xx} - 2u_{xy} + 3u_{yy} + 24u_y + 5u = 0$

Comparing this equation with the general form of a second-order PDE, $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 1$, $B = -2$, $C = 3$, $D = 0$, $E = 24$, $F = 5$, and $G = 0$. The characteristic equations of this PDE are given by

$$\frac{dy}{dx} = \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)$$
$$\frac{dy}{dx} = \frac{1}{2}\left(-2 \pm \sqrt{4 - 12}\right)$$
$$\frac{dy}{dx} = \frac{1}{2}\left(-2 \pm 2i\sqrt{2}\right)$$
$$\frac{dy}{dx} = -1 \pm i\sqrt{2}.$$

Note that the discriminant, $B^2 - 4AC = 4 - 12 = -8$, is less than 0, which means that the PDE is **elliptic** for all x and y . Therefore, the solutions to the ordinary differential equations are two distinct families of characteristic curves that lie in the complex plane. Integrating the characteristic equations, we find that

$$y(x) = \left(-1 \pm i\sqrt{2}\right) x + C_0.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\text{Working with } -i\sqrt{2}: \quad C_0 = y + x + ix\sqrt{2} = \phi(x, y)$$
$$\text{Working with } +i\sqrt{2}: \quad C_0 = y + x - ix\sqrt{2} = \psi(x, y).$$

Since $\xi = \phi(x, y) = y + x + ix\sqrt{2}$ and $\eta = \psi(x, y) = y + x - ix\sqrt{2}$ are complex conjugates of one another, we introduce the new real variables,

$$\alpha = \frac{\xi + \eta}{2} = y + x$$
$$\beta = \frac{\xi - \eta}{2i} = x\sqrt{2},$$

which transform the PDE to the canonical form. After changing variables $(x, y) \to (\alpha, \beta)$, the PDE becomes

$$A^{**}u_{\alpha\alpha} + B^{**}u_{\alpha\beta} + C^{**}u_{\beta\beta} + D^{**}u_\alpha + E^{**}u_\beta + F^{**}u = G^{**},$$

where, using the chain rule,

$$A^{**} = A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2$$
$$B^{**} = 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y$$
$$C^{**} = A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2$$
$$D^{**} = A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y$$
$$E^{**} = A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y$$
$$F^{**} = F$$
$$G^{**} = G.$$

Plugging in the numbers and derivatives to these equations, we find that $A^{**} = 2$, $B^{**} = 0$, $C^{**} = 2$, $D^{**} = 24$, $E^{**} = 0$, $F^{**} = 5$, and $G^{**} = 0$. Thus, the PDE simplifies to

$$2u_{\alpha\alpha} + 2u_{\beta\beta} + 24u_\alpha + 5u = 0.$$

Solving for $u_{\alpha\alpha} + u_{\beta\beta}$ gives

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{2}(24u_\alpha + 5u).$$

This is the canonical form of the elliptic PDE.

Part (d)

$u_{xx} + \operatorname{sech}^4 x u_{yy} = 0$

Comparing this equation with the general form of a second-order PDE, $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 1$, $B = 0$, $C = \operatorname{sech}^4 x$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right) \\ \frac{dy}{dx} &= \frac{1}{2}\left(\pm\sqrt{-4\operatorname{sech}^4 x}\right) \\ \frac{dy}{dx} &= \pm i \operatorname{sech}^2 x. \end{aligned}$$

Note that the discriminant, $B^2 - 4AC = -4\operatorname{sech}^4 x$, is less than 0 for all x, which means that the PDE is **elliptic**. Therefore, the solutions to the ordinary differential equations are two distinct families of characteristic curves that lie in the complex plane. Integrating the characteristic equations, we find that

$$y(x) = \pm i \tanh x + C_0$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\begin{aligned} &\text{Working with } -i\tanh x\text{:} \quad C_0 = y + i\tanh x = \phi(x, y) \\ &\text{Working with } +i\tanh x\text{:} \quad C_0 = y - i\tanh x = \psi(x, y). \end{aligned}$$

Since $\xi = \phi(x, y) = y + i\tanh x$ and $\eta = \psi(x, y) = y - i\tanh x$ are complex conjugates of one another, we introduce the new real variables,

$$\begin{aligned} \alpha &= \frac{1}{2}(\xi + \eta) = y \\ \beta &= \frac{1}{2i}(\xi - \eta) = \tanh x, \end{aligned}$$

which transform the PDE to the canonical form. After changing variables $(x, y) \to (\alpha, \beta)$, the PDE becomes

$$A^{**}u_{\alpha\alpha} + B^{**}u_{\alpha\beta} + C^{**}u_{\beta\beta} + D^{**}u_\alpha + E^{**}u_\beta + F^{**}u = G^{**},$$

where, using the chain rule,

$$\begin{aligned} A^{**} &= A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2 \\ B^{**} &= 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y \\ C^{**} &= A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2 \\ D^{**} &= A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y \\ E^{**} &= A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y \\ F^{**} &= F \\ G^{**} &= G. \end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^{**} = \operatorname{sech}^4 x = \left(\beta^2 - 1\right)^2$, $B^{**} = 0$, $C^{**} = \operatorname{sech}^4 x = \left(\beta^2 - 1\right)^2$, $D^{**} = 0$, $E^{**} = -2\operatorname{sech}^2 x \tanh x = 2\beta\left(\beta^2 - 1\right)$, $F^{**} = 0$, and $G^{**} = 0$. Thus, the PDE simplifies to

$$\left(\beta^2 - 1\right)^2 u_{\alpha\alpha} + \left(\beta^2 - 1\right)^2 u_{\beta\beta} + 2\beta\left(\beta^2 - 1\right)u_\beta = 0$$

$$u_{\alpha\alpha} + u_{\beta\beta} + \frac{2\beta}{\beta^2 - 1} u_\beta = 0.$$

Solving for $u_{\alpha\alpha} + u_{\beta\beta}$ gives

$$u_{\alpha\alpha} + u_{\beta\beta} = \frac{2\beta}{1 - \beta^2} u_\beta.$$

This is the canonical form of the elliptic PDE.

Part (e)

$u_{xx} + 6yu_{xy} + 9y^2u_{yy} + 4u = 0$

Comparing this equation with the general form of a second-order PDE, $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 1$, $B = 6y$, $C = 9y^2$, $D = 0$, $E = 0$, $F = 4$, and $G = 0$. The characteristic equations of this PDE are given by

$$\frac{dy}{dx} = \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)$$
$$\frac{dy}{dx} = \frac{1}{2}\left(6y \pm \sqrt{36y^2 - 36y^2}\right)$$
$$\frac{dy}{dx} = 3y.$$

Note that the discriminant, $B^2 - 4AC = 36y^2 - 36y^2$, is equal to 0 for all y, which means that the PDE is **parabolic**. Therefore, there is one family of real characteristic curves in the xy-plane. Integrating the characteristic equation, we find that

$$\ln|y| = 3x + C_0.$$

So the characteristic curves are given by

$$y(x) = A_0e^{3x}.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$C_0 = \ln|y| - 3x = \phi(x, y).$$

Now we make the change of variables, $\xi = \phi(x, y) = \ln|y| - 3x$. η can be chosen arbitrarily so long as the Jacobian of ξ and η is nonzero. We choose $\eta = y$ for simplicity. With these new variables the PDE becomes

$$A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} + D^*u_\xi + E^*u_\eta + F^*u = G^*,$$

where, using the chain rule, (see page 11 of the textbook for details)

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
F^* &= F \\
G^* &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = 0$, $C^* = 9y^2 = 9\eta^2$, $D^* = -9$, $E^* = 0$, $F^* = 4$, and $G^* = 0$. Thus, the PDE simplifies to

$$9\eta^2u_{\eta\eta} - 9u_\eta + 4u = 0$$
$$u_{\eta\eta} = \frac{1}{9\eta^2}(9u_\eta - 4u).$$

This is the canonical form of the parabolic PDE.

Part (f)

$u_{xx} - \operatorname{sech}^4 x u_{yy} = 0$

Comparing this equation with the general form of a second-order PDE, $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 1$, $B = 0$, $C = -\operatorname{sech}^4 x$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)\\ \frac{dy}{dx} &= \frac{1}{2}\left(\pm\sqrt{4\operatorname{sech}^4 x}\right)\\ \frac{dy}{dx} &= \pm\operatorname{sech}^2 x.\end{aligned}$$

Note that the discriminant, $B^2 - 4AC = 4\operatorname{sech}^4 x$, is greater than 0 for all x, which means that the PDE is **hyperbolic**. The two families of characteristic curves, therefore, are distinct and lie in the xy-plane. Integrating the characteristic equations, we find that

$$y(x) = \pm\tanh x + C_0.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\begin{aligned}&\text{Working with } -\tanh x: \quad C_0 = y + \tanh x = \phi(x, y)\\ &\text{Working with } +\tanh x: \quad C_0 = y - \tanh x = \psi(x, y).\end{aligned}$$

Make the change of variables, $\xi = \phi(x, y) = y + \tanh x$ and $\eta = \psi(x, y) = y - \tanh x$, so that the PDE takes the simplest form. Solving these two equations for x and y gives $x = \tanh^{-1}\left[\frac{1}{2}(\xi - \eta)\right]$ and $y = \frac{1}{2}(\xi + \eta)$. With these new variables the PDE becomes

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*,$$

where, using the chain rule, (see page 11 of the textbook for details)

$$\begin{aligned}A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2\\ B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y\\ C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2\\ D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y\\ E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y\\ F^* &= F\\ G^* &= G.\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $C^* = 0$, $F^* = 0$, $G^* = 0$,

$$\begin{aligned}B^* &= -4\operatorname{sech}^4 x = -\frac{1}{4}\left[(\xi - \eta)^2 - 4\right]^2,\\ D^* &= -2\operatorname{sech}^2 x \tanh x = \frac{1}{4}(\xi - \eta)\left[(\xi - \eta)^2 - 4\right],\\ E^* &= 2\operatorname{sech}^2 x \tanh x = -\frac{1}{4}(\xi - \eta)\left[(\xi - \eta)^2 - 4\right].\end{aligned}$$

Thus, the PDE simplifies to

$$-\frac{1}{4}\left[(\xi-\eta)^2-4\right]^2 u_{\xi\eta}+\frac{1}{4}(\xi-\eta)\left[(\xi-\eta)^2-4\right](u_\xi-u_\eta)=0$$

$$u_{\xi\eta}+\frac{\xi-\eta}{4-(\xi-\eta)^2}(u_\xi-u_\eta)=0$$

$$u_{\xi\eta}=\frac{\xi-\eta}{(\xi-\eta)^2-4}(u_\xi-u_\eta).$$

This is the first canonical form of the hyperbolic PDE. To get the second canonical form, we have to make the additional change of variables, $\alpha=\xi+\eta$ and $\beta=\xi-\eta$. The chain rule gives $u_{\xi\eta}=u_{\alpha\alpha}-u_{\beta\beta}$, $u_\xi=u_\alpha+u_\beta$, and $u_\eta=u_\alpha-u_\beta$. The PDE becomes

$$u_{\alpha\alpha}-u_{\beta\beta}=\frac{2\beta}{\beta^2-4}u_\beta.$$

This is the second canonical form of the hyperbolic PDE.

Part (g)

$u_{xx} + 2\csc y u_{xy} + \csc^2 y u_{yy} = 0$

Comparing this equation with the general form of a second-order PDE, $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 1$, $B = 2\csc y$, $C = \csc^2 y$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right) \\
\frac{dy}{dx} &= \frac{1}{2}\left(2\csc y \pm \sqrt{4\csc^2 y - 4\csc^2 y}\right) \\
\frac{dy}{dx} &= \csc y = \frac{1}{\sin y}.
\end{aligned}$$

Note that the discriminant, $B^2 - 4AC = 4\csc^2 y - 4\csc^2 y$, is equal to 0 for all y, which means that the PDE is **parabolic**. Therefore, there is one family of characteristic curves in the xy-plane. Separating variables and integrating the characteristic equation, we find that

$$-\cos y = x + C_0,$$

and the characteristic curves are given by

$$y(x) = \cos^{-1}(-x - C_0).$$

Solving for the constant of integration (or any convenient multiple thereof),

$$-C_0 = x + \cos y = \phi(x, y).$$

Now we make the change of variables, $\xi = \phi(x, y) = x + \cos y$. η can be chosen arbitrarily so long as the Jacobian of ξ and η is nonzero. We choose $\eta = y$ for simplicity. With these new variables the PDE becomes

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*,$$

where, using the chain rule, (see page 11 of the textbook for details)

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
F^* &= F \\
G^* &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = 0$, $C^* = \csc^2 y = \csc^2 \eta$, $D^* = -\cot y \csc y = -\cot \eta \csc \eta$, $E^* = 0$, $F^* = 0$, and $G^* = 0$. Thus, the PDE simplifies to

$$(\csc^2 \eta)u_{\eta\eta} - (\cot \eta \csc \eta)u_\xi = 0$$

$$\frac{1}{\sin^2 \eta} u_{\eta\eta} - \frac{\cos \eta}{\sin \eta}\frac{1}{\sin \eta} u_\xi = 0$$

$$u_{\eta\eta} = \frac{\cot\eta}{\csc\eta} u_\xi$$

$$u_{\eta\eta} = (\cos\eta) u_\xi.$$

This is the canonical form of the parabolic PDE.

This answer is in disagreement with the answer at the back of the book–there is an extra $\sin^2\eta$ term on the right. I believe the book is in error.

Part (h)

$u_{xx} - 5u_{xy} + 5u_{yy} = 0$

Comparing this equation with the general form of a second-order PDE, $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 1$, $B = -5$, $C = 5$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\frac{dy}{dx} = \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)$$
$$\frac{dy}{dx} = \frac{1}{2}\left(-5 \pm \sqrt{25 - 20}\right)$$
$$\frac{dy}{dx} = \frac{1}{2}\left(-5 \pm \sqrt{5}\right).$$

Note that the discriminant, $B^2 - 4AC = 25 - 20$, is equal to 5, which means that the PDE is **hyperbolic**. The two families of characteristic curves, therefore, are distinct and lie in the xy-plane. Integrating the characteristic equations, we find that

$$y(x) = \frac{1}{2}\left(-5 \pm \sqrt{5}\right)x + C_0.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\text{Working with } -\sqrt{5}: \quad 2C_0 = 2y + \left(5 + \sqrt{5}\right)x = \phi(x, y)$$
$$\text{Working with } +\sqrt{5}: \quad 2C_0 = 2y + \left(5 - \sqrt{5}\right)x = \psi(x, y).$$

Now we make the change of variables, $\xi = \phi(x, y) = 2y + \left(5 + \sqrt{5}\right)x$ and $\eta = \psi(x, y) = 2y + \left(5 - \sqrt{5}\right)x$, so that the PDE takes the simplest form. With these new variables the PDE becomes

$$A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} + D^*u_\xi + E^*u_\eta + F^*u = G^*,$$

where, using the chain rule, (see page 11 of the textbook for details)

$$A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$$
$$B^* = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$
$$C^* = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$$
$$D^* = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y$$
$$E^* = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y$$
$$F^* = F$$
$$G^* = G.$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = -20$, $C^* = 0$, $D^* = 0$, $E^* = 0$, $F^* = 0$, and $G^* = 0$. Thus, the PDE simplifies to

$$-20u_{\xi\eta} = 0$$
$$u_{\xi\eta} = 0.$$

This is the first canonical form of the hyperbolic PDE. If we make the additional change of variables, $\alpha = \xi + \eta$ and $\beta = \xi - \eta$, then the chain rule gives $u_{\xi\eta} = u_{\alpha\alpha} - u_{\beta\beta}$, $u_\xi = u_\alpha + u_\beta$, and $u_\eta = u_\alpha - u_\beta$. The PDE then becomes

$$u_{\alpha\alpha} - u_{\beta\beta} = 0.$$

This is the second canonical form of the hyperbolic PDE.