Exercise 4

(a) Show that the nonlinear equation

$$u^2 u_{xx} + 2u_x u_y u_{xy} - u^2 u_{yy} = 0$$

is hyperbolic for every solution $u(x, y)$.

(b) Show that the nonlinear equation for the velocity potential $u(x, y)$

$$(1 - u_x^2)u_{xx} - 2u_x u_y u_{xy} + (1 - u_y^2)u_{yy} = 0$$

in certain kinds of compressible fluid flow is (i) elliptic, (ii) parabolic, or (iii) hyperbolic for those solutions such that $|\nabla u| < 1$, $|\nabla u| = 1$, or $|\nabla u| > 1$.

Solution

Part (a)

$u^2 u_{xx} + 2u_x u_y u_{xy} - u^2 u_{yy} = 0$

Comparing this equation with the general form of a second-order PDE,
$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = u^2$, $B = 2u_x u_y$, $C = -u^2$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. To show the equation is hyperbolic, we have to consider the discriminant, $B^2 - 4AC$:

$$\begin{aligned} B^2 - 4AC &= (2u_x u_y)^2 - 4u^2(-u^2) \\ &= (2u_x u_y)^2 + 4u^4 \\ &= \underbrace{(2u_x u_y)^2}_{\geq 0} + \underbrace{(2u^2)^2}_{> 0}. \end{aligned}$$

Since the discriminant is a sum of squares, it must be greater than 0. Therefore, the PDE is **hyperbolic** for all $u(x, y)$.

Part (b)

$(1 - u_x^2)u_{xx} - 2u_x u_y u_{xy} + (1 - u_y^2)u_{yy} = 0$

In this case, $A = 1 - u_x^2$, $B = -2u_x u_y$, $C = 1 - u_y^2$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The discriminant is given by

$$\begin{aligned} B^2 - 4AC &= 4u_x^2 u_y^2 - 4(1 - u_x^2)(1 - u_y^2) \\ &= 4(u_x^2 + u_y^2 - 1) \end{aligned}$$

$B^2 - 4AC = 4(u_x^2 + u_y^2 - 1)$, can be positive, zero, or negative, depending on whether $u_x^2 + u_y^2 - 1 > 0$, $u_x^2 + u_y^2 - 1 = 0$, or $u_x^2 + u_y^2 - 1 < 0$, respectively. That is,

$$\text{The PDE is} \begin{cases} \text{hyperbolic} & \text{if } u_x^2 + u_y^2 > 1. \\ \text{parabolic} & \text{if } u_x^2 + u_y^2 = 1. \\ \text{elliptic} & \text{if } u_x^2 + u_y^2 < 1. \end{cases}$$

Note that $u_x^2 + u_y^2 = |\nabla u|^2$, and if we take the square root of both sides of each condition, we get the desired result.

$$\text{The PDE is} \begin{cases} \text{hyperbolic} & \text{if } |\nabla u| > 1. \\ \text{parabolic} & \text{if } |\nabla u| = 1. \\ \text{elliptic} & \text{if } |\nabla u| < 1. \end{cases}$$