Exercise 5

Solve Example 1.6.2 with the boundary conditions

$$u_x(0,t) = 0 = u_x(\ell,t) \quad \text{for } t > 0,$$

leaving the initial condition (1.6.38) unchanged.

Solution

The temperature distribution $u(x,t)$ in a homogeneous rod of length l with diffusivity constant κ satisfies the following initial boundary value problem:

$$\begin{array}{ll} u_t = \kappa u_{xx}, & 0 < x < \ell,\ t > 0 \\ u_x(0,t) = 0, & t > 0 \\ u_x(\ell,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 \leq x \leq \ell. \end{array}$$

The PDE and the boundary conditions are linear and homogeneous, which means that the method of separation of variables can be applied. Assume a product solution of the form, $u(x,t) = X(x)T(t)$, and substitute it into the PDE and boundary conditions:

$$X(x)T'(t) = \kappa X''(x)T(t) \quad \rightarrow \quad \frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)} = k \tag{1.5.1}$$

$$\begin{aligned} u_x(0,t) = 0 \quad &\rightarrow \quad X'(0)T(t) = 0 \quad \rightarrow \quad X'(0) = 0 \\ u_x(\ell,t) = 0 \quad &\rightarrow \quad X'(\ell)T(t) = 0 \quad \rightarrow \quad X'(\ell) = 0. \end{aligned}$$

The left side of equation (1.5.1) is a function of t, and the right side is a function of x. Therefore, both sides must be equal to a constant. Values of this constant and the corresponding functions that satisfy the boundary conditions are known as eigenvalues and eigenfunctions, respectively. We have to examine three special cases: the case where the eigenvalues are positive ($k = \mu^2$), the case where the eigenvalue is zero ($k = 0$), and the case where the eigenvalues are negative ($k = -\lambda^2$). The solution to the PDE will be a linear combination of all product solutions.

Case I: Consider the Positive Eigenvalues $(k = \mu^2)$

Solving the ordinary differential equation in (1.5.1) for $X(x)$ gives

$$X''(x) = \mu^2 X(x), \quad X'(0) = 0,\ X'(\ell) = 0.$$

$$\begin{aligned} X(x) &= C_1 \cosh \mu x + C_2 \sinh \mu x \\ X'(x) &= C_1 \mu \sinh \mu x + C_2 \mu \cosh \mu x \\ X'(0) &= C_2 \mu = 0 \quad \rightarrow \quad C_2 = 0 \\ X'(\ell) &= C_1 \mu \sinh \mu\ell = 0 \quad \rightarrow \quad C_1 = 0 \\ X(x) &= 0 \end{aligned}$$

Positive values of k lead to the trivial solution, $X(x) = 0$. Therefore, there are no positive eigenvalues and no associated product solutions.

Case II: Consider the Zero Eigenvalue $(k = 0)$

Solving the ordinary differential equation for $X(x)$ in (1.5.1) gives

$$\begin{aligned}
&X''(x) = 0, \quad X'(0) = 0,\ X'(\ell) = 0.\\
&X(x) = C_1 x + C_2\\
&X'(x) = C_1\\
&X'(0) = X'(\ell) = C_1 \quad \rightarrow \quad C_1 = 0\\
&X(x) = C_2
\end{aligned}$$

$k = 0$ leads to a nontrivial solution for $X(x)$, so zero is an eigenvalue. Solving the ordinary differential equation for $T(t)$, $T'(t) = 0$, gives $T(t) = C_3$. The product solution associated with the zero eigenvalue is thus a constant.

Case III: Consider the Negative Eigenvalues $(k = -\lambda^2)$

Solving the ordinary differential equation for $X(x)$ in (1.5.1) gives

$$X''(x) = -\lambda^2 X(x), \quad X'(0) = 0,\ X'(\ell) = 0.$$

$$\begin{aligned}
X(x) &= C_1 \cos \lambda x + C_2 \sin \lambda x\\
X'(x) &= -C_1 \lambda \sin \lambda x + C_2 \lambda \cos \lambda x\\
X'(0) &= C_2 \lambda = 0 \quad \rightarrow \quad C_2 = 0\\
X'(\ell) &= -C_1 \lambda \sin \lambda \ell = 0\\
&\qquad \sin \lambda \ell = 0 \quad \rightarrow \quad \lambda \ell = n\pi,\ n = 1, 2, \ldots\\
X(x) &= C_1 \cos \lambda x \qquad\qquad \lambda_n = \frac{n\pi}{\ell},\ n = 1, 2, \ldots
\end{aligned}$$

The eigenvalues are $k = -\lambda_n^2 = -\left(\frac{n\pi}{\ell}\right)^2$, and the corresponding eigenfunctions are $X_n(x) = \cos \frac{n\pi x}{\ell}$. Solving the ordinary differential equation for $T(t)$, $T'(t) = -\kappa \lambda^2 T(t)$, gives $T(t) = C_3 e^{-\kappa \lambda^2 t}$. The product solutions associated with the negative eigenvalues are thus $u_n(x,t) = X_n(x) T_n(t) = e^{-\kappa \lambda_n^2 t} \cos \frac{n\pi x}{\ell}$ for $n = 1, 2, \ldots$.

According to the principle of superposition, the solution to the PDE is a linear combination of all product solutions:

$$\boxed{u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\kappa \left(\frac{n\pi}{\ell}\right)^2 t} \cos \frac{n\pi x}{\ell}.}$$

The coefficients, A_0 and A_n, are determined from the initial condition, so set $t = 0$:

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{\ell} = f(x). \tag{1.5.2}$$

To determine A_0, simply integrate both sides of equation (1.5.2) with respect to x from 0 to ℓ.

$$\int_0^\ell A_0\, dx + \int_0^\ell \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{\ell}\, dx = \int_0^\ell f(x)\, dx$$

$$A_0 \ell + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^\ell \cos \frac{n\pi x}{\ell}\, dx}_{=0} = \int_0^\ell f(x)\, dx$$

$$\boxed{A_0 = \frac{1}{\ell}\int_0^\ell f(x)\,dx}$$

To determine A_n, multiply both sides of equation (1.5.2) by $\cos\frac{m\pi x}{\ell}$ and integrate both sides with respect to x from 0 to ℓ. (m is a positive integer.)

$$A_0\cos\frac{m\pi x}{\ell} + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{\ell}\cos\frac{m\pi x}{\ell} = f(x)\cos\frac{m\pi x}{\ell}$$

$$\int_0^\ell A_0\cos\frac{m\pi x}{\ell}\,dx + \int_0^\ell \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{\ell}\cos\frac{m\pi x}{\ell}\,dx = \int_0^\ell f(x)\cos\frac{m\pi x}{\ell}\,dx$$

$$A_0\underbrace{\int_0^\ell \cos\frac{m\pi x}{\ell}\,dx}_{=0} + \sum_{n=1}^{\infty} A_n\underbrace{\int_0^\ell \cos\frac{n\pi x}{\ell}\cos\frac{m\pi x}{\ell}\,dx}_{=\frac{\ell}{2}\delta_{nm}} = \int_0^\ell f(x)\cos\frac{m\pi x}{\ell}\,dx$$

$$A_n\frac{\ell}{2} = \int_0^\ell f(x)\cos\frac{n\pi x}{\ell}\,dx$$

$$\boxed{A_n = \frac{2}{\ell}\int_0^\ell f(x)\cos\frac{n\pi x}{\ell}\,dx}$$

It is thanks to the orthogonality of the trigonometric functions that most terms in the infinite series vanish upon integration. Only the $n = m$ term remains, and this is denoted by the Kronecker delta function,

$$\delta_{nm} = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}.$$