Exercise 6

Use the separation of variables to solve the Laplace equation

$$u_{xx} + u_{yy} = 0, \quad 0 \le x \le a,\ 0 \le y \le b,$$

with $u(0, y) = 0 = u(a, y)$ for $0 \le y \le b$, and $u(x, 0) = f(x)$ for $0 < x < a$; $u(x, b) = 0$ for $0 \le x \le a$.

Solution

The PDE and the boundary conditions are linear and homogeneous, which means that the method of separation of variables can be applied. Assume a product solution of the form, $u(x, y) = X(x)Y(y)$, and substitute it into the PDE and boundary conditions:

$$X''(x)Y(y) + X(x)Y''(y) = 0 \quad \rightarrow \quad \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = k. \tag{1.6.1}$$

$$\begin{aligned} u(0, y) = 0 \quad &\rightarrow \quad X(0)Y(y) = 0 \quad \rightarrow \quad X(0) = 0 \\ u(a, y) = 0 \quad &\rightarrow \quad X(a)Y(y) = 0 \quad \rightarrow \quad X(a) = 0 \\ u(x, b) = 0 \quad &\rightarrow \quad X(x)Y(b) = 0 \quad \rightarrow \quad Y(b) = 0 \end{aligned}$$

The left side of equation (1.6.1) is a function of x, and the right side is a function of y. Therefore, both sides must be equal to a constant. Values of this constant and the corresponding functions that satisfy the boundary conditions are known as eigenvalues and eigenfunctions, respectively. We have to examine three special cases: the case where the eigenvalues are positive ($k = \mu^2$), the case where the eigenvalue is zero ($k = 0$), and the case where the eigenvalues are negative ($k = -\lambda^2$). The solution to the PDE will be a linear combination of all product solutions. Note that it doesn't matter what side of equation (1.6.1) the minus sign is placed so long as all eigenvalues are accounted for.

Case I: Consider the Positive Eigenvalues ($k = \mu^2$)

Solving the ordinary differential equation in (1.6.1) for $X(x)$ gives

$$X''(x) = \mu^2 X(x), \quad X(0) = 0,\ X(a) = 0.$$

$$\begin{aligned} &X(x) = C_1 \cosh \mu x + C_2 \sinh \mu x \\ &X(0) = C_1 \quad \rightarrow \quad C_1 = 0 \\ &X(a) = C_2 \sinh \mu a = 0 \quad \rightarrow \quad C_2 = 0 \\ &X(x) = 0 \end{aligned}$$

Positive values of k lead to the trivial solution, $X(x) = 0$. Therefore, there are no positive eigenvalues and no associated product solutions.

Case II: Consider the Zero Eigenvalue ($k = 0$)

Solving the ordinary differential equation in (1.6.1) for $X(x)$ gives

$$X''(x) = 0, \quad X(0) = 0,\ X(a) = 0.$$

$$\begin{aligned} X(x) &= C_1 x + C_2 \\ X(0) &= C_2 \quad \to \quad C_2 = 0 \\ X(a) &= C_1 a = 0 \quad \to \quad C_1 = 0 \\ X(x) &= 0 \end{aligned}$$

$k = 0$ leads to the trivial solution, $X(x) = 0$. Therefore, zero is not an eigenvalue, and there's no product solution associated with it.

Case III: Consider the Negative Eigenvalues $(k = -\lambda^2)$

Solving the ordinary differential equation in (1.6.1) for $X(x)$ gives

$$X''(x) = -\lambda^2 X(x), \quad X(0) = 0, \ X(a) = 0.$$

$$\begin{aligned} X(x) &= C_1 \cos \lambda x + C_2 \sin \lambda x \\ X(0) &= C_1 \quad \to \quad C_1 = 0 \\ X(a) &= C_2 \sin \lambda a = 0 \\ &\quad \sin \lambda a = 0 \quad \to \quad \lambda a = n\pi, \ n = 1, 2, \ldots \\ X(x) &= C_2 \sin \lambda x \qquad\qquad \lambda_n = \frac{n\pi}{a}, \ n = 1, 2, \ldots \end{aligned}$$

The eigenvalues are $k = -\lambda_n^2 = -\left(\frac{n\pi}{a}\right)^2$, and the corresponding eigenfunctions are $X_n(x) = \sin \frac{n\pi x}{a}$. Solving the ordinary differential equation for $Y(y)$, $Y''(y) = \lambda^2 Y(y)$, with $Y(b) = 0$ gives $Y(y) = C_3 \sinh \lambda(b - y)$. The product solutions associated with the negative eigenvalues are thus $u_n(x, y) = X_n(x) Y_n(y) = \sinh \frac{n\pi(b-y)}{a} \sin \frac{n\pi x}{a}$ for $n = 1, 2, \ldots$.

According to the principle of superposition, the solution to the PDE is a linear combination of all product solutions:

$$\boxed{u(x, y) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi(b-y)}{a} \sin \frac{n\pi x}{a}.}$$

The coefficients, B_n, are determined from the nonzero boundary condition,

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a} = f(x). \tag{1.6.2}$$

To determine B_n, multiply both sides of equation (1.6.2) by $\sin \frac{m\pi x}{a}$ and integrate both sides with respect to x from 0 to a. (m is a positive integer.)

$$\sum_{n=1}^{\infty} B_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a} \sin \frac{m\pi x}{a} = f(x) \sin \frac{m\pi x}{a}$$

$$\int_0^a \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a} \sin \frac{m\pi x}{a} \, dx = \int_0^a f(x) \sin \frac{m\pi x}{a} \, dx$$

$$\sum_{n=1}^{\infty} B_n \sinh \frac{n\pi b}{a} \underbrace{\int_0^a \sin \frac{n\pi x}{a} \sin \frac{m\pi x}{a} \, dx}_{= \frac{a}{2}\delta_{nm}} = \int_0^a f(x) \sin \frac{m\pi x}{a} \, dx$$

$$B_n \sinh \frac{n\pi b}{a} \left(\frac{a}{2}\right) = \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx$$

$$\boxed{B_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx}$$

It is thanks to the orthogonality of the trigonometric functions that most terms in the infinite series vanish upon integration. Only the $n = m$ term remains, and this is denoted by the Kronecker delta function,

$$\delta_{nm} = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}.$$