Exercise 8

Solve the problem in Exercise 4 [TYPO: Use Exercise 7!] with the boundary conditions

$$\begin{aligned} u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad \text{for } 0 \le x \le \ell, \\ u(0,t) = 0 = u(\ell,t) \quad \text{for } t > 0, \\ u_{xx}(0,t) = 0 = u_{xx}(\ell,t) \quad \text{for } t > 0. \end{aligned}$$

Solution

There is a typo in this problem; one should refer to Exercise 7 rather than Exercise 4 so that the answer obtained matches the one at the back of the book. The initial boundary value problem that needs to be solved is the following:

$$\begin{array}{ll} u_{tt} + c^2 u_{xxxx} = 0, & 0 < x < \ell, \ t > 0 \\ u(0,t) = 0 = u(\ell,t), & t > 0 \\ u_{xx}(0,t) = 0 = u_{xx}(\ell,t), & t > 0 \\ u(x,0) = f(x), & 0 \le x \le \ell \\ u_t(x,0) = g(x), & 0 \le x \le \ell. \end{array}$$

The PDE and the boundary conditions are linear and homogeneous, which means that the method of separation of variables can be applied. Assume a product solution of the form, $u(x,t) = X(x)T(t)$, and substitute it into the PDE and boundary conditions to obtain

$$X(x)T''(t) + c^2 X''''(x)T(t) = 0 \quad \to \quad -\frac{T''(t)}{c^2 T(t)} = \frac{X''''(x)}{X(x)} = k \tag{1}$$

$$\begin{aligned} u(0,t) = 0 \quad &\to \quad X(0)T(t) = 0 \quad \to \quad X(0) = 0 \\ u(\ell,t) = 0 \quad &\to \quad X(\ell)T(t) = 0 \quad \to \quad X(\ell) = 0 \\ u_{xx}(0,t) = 0 \quad &\to \quad X''(0)T(t) = 0 \quad \to \quad X''(0) = 0 \\ u_{xx}(\ell,t) = 0 \quad &\to \quad X''(\ell)T(t) = 0 \quad \to \quad X''(\ell) = 0 \end{aligned}$$

The left side of (1) is a function of t, and the right side is a function of x. Therefore, both sides must be equal to a constant. This constant must be positive so that the solution to $T''(t) = -kc^2 T(t)$ remains finite as $t \to \infty$. The constant is not zero because it would only yield the trivial solution. Let $k = \lambda^4$; the reason for choosing λ^4 instead of λ^2 is to make the equation for $X(x)$ more convenient to solve.

$$\frac{d^4 X}{dx^4} - \lambda^4 X = 0, \quad X(0) = 0, \ X(\ell) = 0, \ X''(0) = 0, \ X''(\ell) = 0$$

This is a linear homogeneous ordinary differential equation with constant coefficients, so the solution has the form, $X(x) = e^{rx}$. Substituting this into the equation gives

$$\begin{aligned} &r^4 e^{rx} - \lambda^4 e^{rx} = 0 \\ &e^{rx}(r^4 - \lambda^4) = 0 \\ &r^4 - \lambda^4 = 0 \\ &(r^2 + \lambda^2)(r^2 - \lambda^2) = 0 \\ &(r + i\lambda)(r - i\lambda)(r + \lambda)(r - \lambda) = 0 \\ &\to \quad r = \{\pm i\lambda, \pm\lambda\} \end{aligned}$$

$X(x)$ is simply a linear combination of the e^{rx} terms:

$$X(x) = D_1 e^{-i\lambda x} + D_2 e^{i\lambda x} + D_3 e^{-\lambda x} + D_4 e^{\lambda x}.$$

If we set the constants to be $D_1 = \frac{1}{2}(iC_1 + C_2)$, $D_2 = \frac{1}{2}(-iC_1 + C_2)$, $D_3 = \frac{1}{2}(C_3 - C_4)$, and $D_4 = \frac{1}{2}(C_3 + C_4)$, then we can rewrite $X(x)$ in terms of trigonometric functions. Recall that

$$\begin{aligned}
\sin x &= \frac{1}{2i}(e^{ix} - e^{-ix}) \\
\cos x &= \frac{1}{2}(e^{ix} + e^{-ix}) \\
\sinh x &= \frac{1}{2}(e^{x} - e^{-x}) \\
\cosh x &= \frac{1}{2}(e^{x} + e^{-x}).
\end{aligned}$$

So we have

$$X(x) = C_1 \sin \lambda x + C_2 \cos \lambda x + C_3 \sinh \lambda x + C_4 \cosh \lambda x.$$

Now we apply the boundary conditions to determine the constants.

$$\begin{aligned}
X(0) &= C_2 + C_4 = 0 \\
X''(0) &= \lambda^2(-C_2 + C_4) = 0 \\
X(\ell) &= C_1 \sin \lambda\ell + C_3 \sinh \lambda\ell = 0 \\
X''(\ell) &= \lambda^2(-C_1 \sin \lambda\ell + C_3 \sinh \lambda\ell = 0
\end{aligned}$$

The first two equations imply that $C_2 = C_4 = 0$. Since $\sinh \lambda\ell$ is greater than zero for all positive λ, set $C_3 = 0$. C_1 may then be arbitrary so long as $\sin \lambda\ell = 0$. This is true when $\lambda\ell = n\pi$ or $\lambda_n = \frac{n\pi}{\ell}$. These are the eigenvalues, and the corresponding eigenfunctions are $X_n(x) = \sin \frac{n\pi x}{\ell}$. Solving the ordinary differential equation for $T(t)$, $T''(t) = -c^2\lambda^4 T(t)$, gives $T(t) = A \cos c\lambda^2 t + B \sin c\lambda^2 t$. The product solutions are thus $u_n(x,t) = X_n(x)T_n(t) = \sin \lambda_n x(A_n \cos c\lambda_n^2 t + B_n \sin c\lambda_n^2 t)$ for $n = 1, 2, \ldots$.

According to the principle of superposition, the solution to the PDE is a linear combination of all product solutions:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos c \left(\frac{n\pi}{\ell}\right)^2 t + B_n \sin c \left(\frac{n\pi}{\ell}\right)^2 t \right] \sin \frac{n\pi x}{\ell}.}$$

The constants A_n and B_n may be determined from the initial conditions of the problem.

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{\ell} = f(x)$$

Multiplying both sides of the equation by $\sin \frac{m\pi x}{\ell}$ (m being a positive integer) gives

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{\ell} \sin \frac{m\pi x}{\ell} = f(x) \sin \frac{m\pi x}{\ell}.$$

Integrating both sides with respect to x from 0 to ℓ gives

$$\int_0^\ell \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{\ell} \sin\frac{m\pi x}{\ell}\,dx = \int_0^\ell f(x)\sin\frac{m\pi x}{\ell}\,dx$$

$$\sum_{n=1}^{\infty} A_n \underbrace{\int_0^\ell \sin\frac{n\pi x}{\ell} \sin\frac{m\pi x}{\ell}\,dx}_{=\frac{\ell}{2}\delta_{nm}} = \int_0^\ell f(x)\sin\frac{m\pi x}{\ell}\,dx$$

$$A_n\left(\frac{\ell}{2}\right) = \int_0^\ell f(x)\sin\frac{n\pi x}{\ell}\,dx$$

$$\boxed{A_n = \frac{2}{\ell}\int_0^\ell f(x)\sin\frac{n\pi x}{\ell}\,dx.}$$

In order to use the second initial condition, we have to take the first derivative of $u(x,t)$ with respect to t.

$$u_t(x,t) = \sum_{n=1}^{\infty}\left[-A_n c\left(\frac{n\pi}{\ell}\right)^2 \sin c\left(\frac{n\pi}{\ell}\right)^2 t + B_n c\left(\frac{n\pi}{\ell}\right)^2 \cos c\left(\frac{n\pi}{\ell}\right)^2 t\right]\sin\frac{n\pi x}{\ell}$$

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n c\left(\frac{n\pi}{\ell}\right)^2 \sin\frac{n\pi x}{\ell} = g(x)$$

Multiplying both sides of the equation by $\sin\frac{m\pi x}{\ell}$ (m being a positive integer) gives

$$\sum_{n=1}^{\infty} B_n c\left(\frac{n\pi}{\ell}\right)^2 \sin\frac{n\pi x}{\ell}\sin\frac{m\pi x}{\ell} = g(x)\sin\frac{m\pi x}{\ell}.$$

Integrating both sides with respect to x from 0 to ℓ gives

$$\int_0^\ell \sum_{n=1}^{\infty} B_n c\left(\frac{n\pi}{\ell}\right)^2 \sin\frac{n\pi x}{\ell}\sin\frac{m\pi x}{\ell}\,dx = \int_0^\ell g(x)\sin\frac{m\pi x}{\ell}\,dx$$

$$\sum_{n=1}^{\infty} B_n c\left(\frac{n\pi}{\ell}\right)^2 \underbrace{\int_0^\ell \sin\frac{n\pi x}{\ell}\sin\frac{m\pi x}{\ell}\,dx}_{=\frac{\ell}{2}\delta_{nm}} = \int_0^\ell g(x)\sin\frac{m\pi x}{\ell}\,dx$$

$$B_n c\left(\frac{n\pi}{\ell}\right)^2\left(\frac{\ell}{2}\right) = \int_0^\ell g(x)\sin\frac{n\pi x}{\ell}\,dx$$

$$\boxed{B_n = \frac{2\ell}{c(n\pi)^2}\int_0^\ell g(x)\sin\frac{n\pi x}{\ell}\,dx.}$$

It is thanks to the orthogonality of the trigonometric functions that most terms in the infinite series vanish upon integration. Only the $n = m$ term remains, and this is denoted by the Kronecker delta function,

$$\delta_{nm} = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}.$$