Exercise 10

Solve Example 1.6.1 with the initial data

$$\text{(i) } f(x) = \begin{cases} \frac{hx}{a} & \text{if } 0 \le x \le a, \\ h(\ell - x)/(\ell - a) & \text{if } a \le x \le \ell, \end{cases} \quad \text{and } g(x) = 0.$$

$$\text{(ii) } f(x) = 0 \quad \text{and} \quad g(x) = \begin{cases} \frac{u_0 x}{a} & \text{if } 0 \le x \le a, \\ u_0(\ell - x)/(\ell - a) & \text{if } a \le x \le \ell. \end{cases}$$

Solution

The initial boundary value problem that needs to be solved is the following:

$$\begin{array}{ll} u_{tt} = c^2 u_{xx}, & 0 < x < \ell,\ t > 0 \\ u(0,t) = u(\ell,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 \le x \le \ell \\ u_t(x,0) = g(x), & 0 \le x \le \ell. \end{array}$$

The PDE and the boundary conditions are linear and homogeneous, which means that the method of separation of variables can be applied. Assume a product solution of the form, $u(x,t) = X(x)T(t)$, and substitute it into the PDE and boundary conditions to obtain

$$X(x)T''(t) = c^2 X''(x)T(t) \quad \rightarrow \quad \frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = k \tag{1.10.1}$$

$$\begin{array}{lllll} u(0,t) = 0 & \rightarrow & X(0)T(t) = 0 & \rightarrow & X(0) = 0 \\ u(\ell,t) = 0 & \rightarrow & X(\ell)T(t) = 0 & \rightarrow & X(\ell) = 0 \end{array}$$

The left side of equation (1.10.1) is a function of t, and the right side is a function of x. Therefore, both sides must be equal to a constant. Values of this constant and the corresponding functions that satisfy the boundary conditions are known as eigenvalues and eigenfunctions, respectively. We have to examine three special cases: the case where the eigenvalues are positive ($k = \mu^2$), the case where the eigenvalue is zero ($k = 0$), and the case where the eigenvalues are negative ($k = -\lambda^2$). The solution to the PDE will be a linear combination of all product solutions.

Case I: Consider the Positive Eigenvalues ($k = \mu^2$)

Solving the ordinary differential equation in (1.10.1) for $X(x)$ gives

$$X''(x) = \mu^2 X(x), \quad X(0) = 0,\ X(\ell) = 0.$$

$$\begin{array}{l} X(x) = C_1 \cosh \mu x + C_2 \sinh \mu x \\ X(0) = C_1 \quad \rightarrow \quad C_1 = 0 \\ X(\ell) = C_2 \sinh \mu\ell = 0 \quad \rightarrow \quad C_2 = 0 \\ X(x) = 0. \end{array}$$

Positive values of k lead to the trivial solution, $X(x) = 0$. Therefore, there are no positive eigenvalues and no associated product solutions.

Case II: Consider the Zero Eigenvalue $(k = 0)$

Solving the ordinary differential equation in (1.10.1) for $X(x)$ gives

$$X''(x) = 0, \quad X(0) = 0, \; X(\ell) = 0.$$

$$\begin{aligned} &X(x) = C_1 x + C_2 \\ &X(0) = C_2 \quad \to \quad C_2 = 0 \\ &X(\ell) = C_1 \ell = 0 \quad \to \quad C_1 = 0 \\ &X(x) = 0. \end{aligned}$$

$k = 0$ leads to the trivial solution, $X(x) = 0$. Therefore, zero is not an eigenvalue, and there's no product solution associated with it.

Case III: Consider the Negative Eigenvalues $(k = -\lambda^2)$

Solving the ordinary differential equation in (1.10.1) for $X(x)$ gives

$$X''(x) = -\lambda^2 X(x), \quad X(0) = 0, \; X(\ell) = 0.$$

$$\begin{aligned} &X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x \\ &X(0) = C_1 \quad \to \quad C_1 = 0 \\ &X(\ell) = C_2 \sin \lambda \ell = 0 \\ &\qquad \sin \lambda \ell = 0 \quad \to \quad \lambda \ell = n\pi, \; n = 1, 2, \ldots \\ &X(x) = C_2 \sin \lambda x \qquad \lambda_n = \frac{n\pi}{\ell}, \; n = 1, 2, \ldots \end{aligned}$$

The eigenvalues are $k = -\lambda_n^2 = -\left(\frac{n\pi}{\ell}\right)^2$, and the corresponding eigenfunctions are $X_n(x) = \sin \frac{n\pi x}{\ell}$. Solving the ordinary differential equation for $T(t)$, $T''(t) = -c^2\lambda^2 T(t)$, gives $T(t) = A \cos c\lambda t + B \sin c\lambda t$. The product solutions associated with the negative eigenvalues are thus $u_n(x, y) = X_n(x)T_n(t) = (A_n \cos c\lambda_n t + B_n \sin c\lambda_n t) \sin \lambda_n x$ for $n = 1, 2, \ldots$.

According to the principle of superposition, the solution to the PDE is a linear combination of all product solutions:

$$u(x, t) = \sum_{n=1}^{\infty} \left(A_n \cos c\frac{n\pi}{\ell} t + B_n \sin c\frac{n\pi}{\ell} t \right) \sin \frac{n\pi}{\ell} x.$$

The coefficients, A_n and B_n, are determined from the initial conditions. Setting $t = 0$ results in an equation for A_n.

$$u(x, 0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{\ell} = f(x)$$

Multiplying both sides of the equation by $\sin \frac{m\pi x}{\ell}$ (m being a positive integer) gives

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{\ell} \sin \frac{m\pi x}{\ell} = f(x) \sin \frac{m\pi x}{\ell}.$$

Integrating both sides with respect to x from 0 to ℓ gives

$$\int_0^{\ell} \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{\ell} \sin \frac{m\pi x}{\ell} \, dx = \int_0^{\ell} f(x) \sin \frac{m\pi x}{\ell} \, dx$$

$$\sum_{n=1}^{\infty} A_n \underbrace{\int_0^\ell \sin\frac{n\pi x}{\ell}\sin\frac{m\pi x}{\ell}\,dx}_{=\frac{\ell}{2}\delta_{nm}} = \int_0^\ell f(x)\sin\frac{m\pi x}{\ell}\,dx$$

$$A_n\left(\frac{\ell}{2}\right) = \int_0^\ell f(x)\sin\frac{n\pi x}{\ell}\,dx$$

$$A_n = \frac{2}{\ell}\int_0^\ell f(x)\sin\frac{n\pi x}{\ell}\,dx.$$

In order to find B_n we have to use the second initial condition, so we take the first derivative of $u(x,t)$ with respect to t.

$$u_t(x,t) = \sum_{n=1}^{\infty}\left(-A_n c\frac{n\pi}{\ell}\sin c\frac{n\pi}{\ell}t + B_n c\frac{n\pi}{\ell}\cos c\frac{n\pi}{\ell}t\right)\sin\frac{n\pi x}{\ell}$$

$$u_t(x,0) = \sum_{n=1}^{\infty}\left(B_n c\frac{n\pi}{\ell}\right)\sin\frac{n\pi x}{\ell} = g(x)$$

Multiplying both sides of the equation by $\sin\frac{m\pi x}{\ell}$ (m being a positive integer) gives

$$\sum_{n=1}^{\infty}\left(B_n c\frac{n\pi}{\ell}\right)\sin\frac{n\pi x}{\ell}\sin\frac{m\pi x}{\ell} = g(x)\sin\frac{m\pi x}{\ell}$$

Integrating both sides with respect to x from 0 to ℓ gives

$$\int_0^\ell \sum_{n=1}^{\infty}\left(B_n c\frac{n\pi}{\ell}\right)\sin\frac{n\pi x}{\ell}\sin\frac{m\pi x}{\ell}\,dx = \int_0^\ell g(x)\sin\frac{m\pi x}{\ell}\,dx$$

$$\sum_{n=1}^{\infty}\left(B_n c\frac{n\pi}{\ell}\right)\underbrace{\int_0^\ell \sin\frac{n\pi x}{\ell}\sin\frac{m\pi x}{\ell}\,dx}_{=\frac{\ell}{2}\delta_{nm}} = \int_0^\ell g(x)\sin\frac{m\pi x}{\ell}\,dx$$

$$\left(B_n c\frac{n\pi}{\ell}\right)\frac{\ell}{2} = \int_0^\ell g(x)\sin\frac{n\pi x}{\ell}\,dx$$

$$B_n = \frac{2}{cn\pi}\int_0^\ell g(x)\sin\frac{n\pi x}{\ell}\,dx$$

Now that we know the general solution of the PDE and the coefficients, we can use the initial data given in the problem statement.

$$\text{(i) } f(x) = \begin{cases} \frac{hx}{a} & \text{if } 0 \le x \le a, \\ h(\ell - x)/(\ell - a) & \text{if } a \le x \le \ell, \end{cases} \quad \text{and } g(x) = 0.$$

The coefficients evaluate to

$$\begin{aligned}
A_n &= \frac{2}{\ell}\int_0^\ell f(x)\sin\frac{n\pi x}{\ell}\,dx \\
&= \frac{2}{\ell}\left(\int_0^a \frac{hx}{a}\sin\frac{n\pi x}{\ell}\,dx + \int_a^\ell \frac{h(\ell-x)}{\ell-a}\sin\frac{n\pi x}{\ell}\,dx\right) \\
&= \frac{2}{\ell}\left\{\left[\frac{h\ell}{an^2\pi^2}\left(\ell\sin\frac{n\pi a}{\ell} - an\pi\cos\frac{n\pi a}{\ell}\right)\right] + \left[\frac{h\ell}{(\ell-a)n^2\pi^2}\left(\ell\sin\frac{n\pi a}{\ell} + (\ell-a)n\pi\cos\frac{n\pi a}{\ell}\right)\right]\right\} \\
&= \frac{2h\ell^2}{a(\ell-a)n^2\pi^2}\sin\frac{n\pi a}{\ell} \\
B_n &= \frac{2}{cn\pi}\int_0^\ell g(x)\sin\frac{n\pi x}{\ell}\,dx \\
&= 0.
\end{aligned}$$

So the solution to the initial boundary value problem (i) is

$$\boxed{u(x,t) = \sum_{n=1}^\infty \frac{2h\ell^2}{a(\ell-a)n^2\pi^2}\sin\frac{n\pi a}{\ell}\cos c\frac{n\pi}{\ell}t\sin\frac{n\pi}{\ell}x.}$$

$$\text{(ii) } f(x) = 0 \quad \text{and} \quad g(x) = \begin{cases} \frac{u_0 x}{a} & \text{if } 0 \le x \le a, \\ u_0(\ell-x)/(\ell-a) & \text{if } a \le x \le \ell. \end{cases}$$

With these initial data the coefficients evaluate to

$$\begin{aligned}
A_n &= \frac{2}{\ell}\int_0^\ell f(x)\sin\frac{n\pi x}{\ell}\,dx \\
&= 0 \\
B_n &= \frac{2}{cn\pi}\int_0^\ell g(x)\sin\frac{n\pi x}{\ell}\,dx \\
&= \frac{2}{cn\pi}\left(\int_0^a \frac{u_0 x}{a}\sin\frac{n\pi x}{\ell}\,dx + \int_a^\ell \frac{u_0(\ell-x)}{\ell-a}\sin\frac{n\pi x}{\ell}\,dx\right) \\
&= \frac{2}{cn\pi}\left\{\left[\frac{u_0\ell}{an^2\pi^2}\left(\ell\sin\frac{n\pi a}{\ell} - an\pi\cos\frac{n\pi a}{\ell}\right)\right] + \left[\frac{u_0\ell}{(\ell-a)n^2\pi^2}\left(\ell\sin\frac{n\pi a}{\ell} + (\ell-a)n\pi\cos\frac{n\pi a}{\ell}\right)\right]\right\} \\
&= \frac{2u_0\ell^3}{ac(\ell-a)n^3\pi^3}\sin\frac{n\pi a}{\ell}.
\end{aligned}$$

So the solution to the initial boundary value problem (ii) is

$$\boxed{u(x,t) = \sum_{n=1}^\infty \frac{2u_0\ell^3}{ac(\ell-a)n^3\pi^3}\sin\frac{n\pi a}{\ell}\sin c\frac{n\pi}{\ell}t\sin\frac{n\pi}{\ell}x.}$$