Exercise 20

Apply the Fourier transform to solve the initial-value problem for the dissipative wave equation

$$u_{tt} = c^2 u_{xx} + \alpha u_{xxt}, \quad -\infty < x < \infty, \ t > 0,$$
$$u(x,0) = f(x), \quad u_t(x,0) = \alpha f''(x) \quad \text{for } -\infty < x < \infty,$$

where α is a positive constant.

Solution

The PDE is defined for $-\infty < x < \infty$, so we can apply the Fourier transform to solve it. We define the Fourier transform here as

$$\mathcal{F}\{u(x,t)\} = U(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u(x,t)\, dx,$$

which means the partial derivatives of u with respect to x and t transform as follows.

$$\mathcal{F}\left\{\frac{\partial^n u}{\partial x^n}\right\} = (ik)^n U(k,t)$$
$$\mathcal{F}\left\{\frac{\partial^n u}{\partial t^n}\right\} = \frac{d^n U}{dt^n}$$

Take the Fourier transform of both sides of the PDE.

$$\mathcal{F}\{u_{tt}\} = \mathcal{F}\{c^2 u_{xx} + \alpha u_{xxt}\}$$

The Fourier transform is a linear operator.

$$\mathcal{F}\{u_{tt}\} = c^2 \mathcal{F}\{u_{xx}\} + \alpha \mathcal{F}\{u_{xxt}\}$$

Transform the derivatives with the relations above.

$$\frac{d^2U}{dt^2} = c^2 (ik)^2 U + \alpha (ik)^2 \frac{dU}{dt}$$

Expand the coefficients.

$$\frac{d^2U}{dt^2} = -c^2 k^2 U - \alpha k^2 \frac{dU}{dt}$$

Bring all terms to the left side.

$$\frac{d^2U}{dt^2} + \alpha k^2 \frac{dU}{dt} + c^2 k^2 U = 0 \tag{1}$$

The PDE has thus been reduced to an ODE. Before we solve it, we have to transform the initial conditions as well. Taking the Fourier transform of the initial conditions gives

$$u(x,0) = f(x) \quad \to \quad \mathcal{F}\{u(x,0)\} = \mathcal{F}\{f(x)\}$$
$$U(k,0) = F(k) \tag{2}$$
$$\frac{\partial u}{\partial t}(x,0) = \alpha f''(x) \quad \to \quad \mathcal{F}\left\{\frac{\partial u}{\partial t}(x,0)\right\} = \mathcal{F}\{\alpha f''(x)\}$$
$$\frac{dU}{dt}(k,0) = \alpha (ik)^2 F(k) = -\alpha k^2 F(k). \tag{3}$$

We can solve equation (1) with the Laplace transform since $t > 0$. The Laplace transform of $U(k, t)$ is defined as

$$\mathcal{L}\{U(k,t)\} = \overline{U}(k,s) = \int_0^\infty e^{-st}U(k,t)\,dt,$$

so the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dU}{dt}\right\} = s\overline{U}(k,s) - U(k,0) \tag{4}$$

$$\mathcal{L}\left\{\frac{d^2U}{dt^2}\right\} = s^2\overline{U}(k,s) - sU(k,0) - \frac{dU}{dt}(k,0) \tag{5}$$

Take the Laplace transform of both sides of equation (1).

$$\mathcal{L}\left\{\frac{d^2U}{dt^2} + \alpha k^2\frac{dU}{dt} + c^2k^2U\right\} = \mathcal{L}\{0\}$$

The Laplace transform is a linear operator.

$$\mathcal{L}\left\{\frac{d^2U}{dt^2}\right\} + \alpha k^2\mathcal{L}\left\{\frac{dU}{dt}\right\} + c^2k^2\mathcal{L}\left\{U\right\} = 0$$

Use equations (4) and (5) here.

$$\left[s^2\overline{U}(k,s) - sU(k,0) - \frac{dU}{dt}(k,0)\right] + \alpha k^2[s\overline{U}(k,s) - U(k,0)] + c^2k^2\overline{U}(k,s) = 0$$

Expand the left side and substitute equations (2) and (3).

$$s^2\overline{U}(k,s) - sF(k) + \cancel{\alpha k^2F(k)} + \alpha k^2s\overline{U}(k,s) - \cancel{\alpha k^2F(k)} + c^2k^2\overline{U}(k,s) = 0$$

The ODE has thus been reduced to an algebraic equation. Factor $\overline{U}(k, s)$ and bring the terms without it to the right side.

$$(s^2 + \alpha k^2s + c^2k^2)\overline{U}(k,s) = sF(k)$$

Divide both sides by $s^2 + \alpha k^2s + c^2k^2$.

$$\overline{U}(k,s) = \frac{s}{s^2 + \alpha k^2s + c^2k^2}F(k)$$

In order to change back to $u(x, t)$, we have to take the inverse Laplace transform of $\overline{U}(k, s)$ to get $U(k, t)$ and then take the inverse Fourier transform of it. Our task now is to write $\overline{U}$ in a form that we can easily transform. The two inverse Laplace transforms we will eventually use are

$$\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2 + b^2}\right\} = e^{at}\cos bt \tag{6}$$

$$\mathcal{L}^{-1}\left\{\frac{b}{(s-a)^2 + b^2}\right\} = e^{at}\sin bt, \tag{7}$$

so we want to write $\overline{U}$ with terms that have these forms. Complete the square in the denominator.

$$\overline{U}(k,s) = \frac{s}{\left(s + \frac{\alpha k^2}{2}\right)^2 + \left(c^2k^2 - \frac{\alpha^2k^4}{4}\right)}F(k)$$

Rewrite $\overline{U}$ so that $s + \alpha k^2/2$ is in the numerator.

$$\overline{U}(k,s) = \frac{s + \frac{\alpha k^2}{2}}{\left(s + \frac{\alpha k^2}{2}\right)^2 + \left(c^2k^2 - \frac{\alpha^2 k^4}{4}\right)} F(k) + \frac{-\frac{\alpha k^2}{2}}{\left(s + \frac{\alpha k^2}{2}\right)^2 + \left(c^2k^2 - \frac{\alpha^2 k^4}{4}\right)} F(k)$$

Multiply the numerator and denominator of the second fraction by $\sqrt{c^2k^2 - \alpha^2k^4/4}$.

$$\overline{U}(k,s) = \frac{s + \frac{\alpha k^2}{2}}{\left(s + \frac{\alpha k^2}{2}\right)^2 + \left(c^2k^2 - \frac{\alpha^2 k^4}{4}\right)} F(k) - \frac{\alpha k^2}{2} \frac{1}{\sqrt{c^2k^2 - \frac{\alpha^2k^4}{4}}} \frac{\sqrt{c^2k^2 - \frac{\alpha^2k^4}{4}}}{\left(s + \frac{\alpha k^2}{2}\right)^2 + \left(c^2k^2 - \frac{\alpha^2 k^4}{4}\right)} F(k)$$

Now we're ready to take the inverse Laplace transform. Use equations (6) and (7) here.

$$U(k,t) = F(k)e^{-\frac{\alpha k^2}{2}t} \cos\left(t\sqrt{c^2k^2 - \frac{\alpha^2k^4}{4}}\right) - \frac{\alpha k^2}{2} \frac{F(k)}{\sqrt{c^2k^2 - \frac{\alpha^2k^4}{4}}} e^{-\frac{\alpha k^2}{2}t} \sin\left(t\sqrt{c^2k^2 - \frac{\alpha^2k^4}{4}}\right)$$

To make $U(k,t)$ easier to work with, introduce a new variable $\omega = \omega(k)$ for the square root term.

$$\omega(k) = \sqrt{c^2k^2 - \frac{\alpha^2k^4}{4}}$$

Then, after factoring,

$$U(k,t) = F(k)e^{-\frac{\alpha k^2}{2}t}\left(\cos\omega t - \frac{\alpha k^2}{2\omega}\sin\omega t\right).$$

It's not necessary to consider the case where $c^2k^2 - \frac{\alpha^2}{4} < 0$ because $\cos i\omega t = \cosh\omega t$ and $-i\sin i\omega t = \sinh\omega t$. We're ready now to take the inverse Fourier transform. It is defined as

$$\mathcal{F}^{-1}\{U(k,t)\} = u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U(k,t)e^{ikx}\,dk.$$

Plug $U(k,t)$ into the definition of the inverse Fourier transform to get $u(x,t)$.

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{-\frac{\alpha k^2}{2}t}\left(\cos\omega t - \frac{\alpha k^2}{2\omega}\sin\omega t\right)e^{ikx}\,dk$$

Recall that sine and cosine can be written in terms of exponentials using Euler's formula.

$$\cos\omega t = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$$
$$\sin\omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$$

Substituting these expressions, we get

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{-\frac{\alpha k^2}{2}t}\left[\left(\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right) - \frac{\alpha k^2}{2\omega}\left(\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right)\right]e^{ikx}\,dk.$$

Expand the integrand and factor the terms in $e^{i\omega t}$ and $e^{-i\omega t}$.

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{-\frac{\alpha k^2}{2}t}\left[\frac{1}{2}\left(1 + \frac{i\alpha k^2}{2\omega}\right)e^{i\omega t} + \frac{1}{2}\left(1 - \frac{i\alpha k^2}{2\omega}\right)e^{-i\omega t}\right]e^{ikx}\,dk$$

Distribute e^{ikx} and $F(k)$ and $1/\sqrt{2\pi}$.

$$u(x,t) = \int_{-\infty}^{\infty} e^{-\frac{\alpha k^2}{2}t}\left[\frac{1}{2}\left(1+\frac{i\alpha k^2}{2\omega}\right)\frac{F(k)}{\sqrt{2\pi}}\,e^{i(kx+\omega t)} + \frac{1}{2}\left(1-\frac{i\alpha k^2}{2\omega}\right)\frac{F(k)}{\sqrt{2\pi}}\,e^{i(kx-\omega t)}\right] dk$$

Therefore,

$$u(x,t) = \int_{-\infty}^{\infty} e^{-\frac{\alpha k^2}{2}t}[A(k)\,e^{i(kx+\omega t)} + B(k)\,e^{i(kx-\omega t)}]\,dk,$$

where

$$\begin{aligned} A(k) &= \frac{1}{2}\left(1+\frac{i\alpha k^2}{2\omega}\right)\frac{F(k)}{\sqrt{2\pi}} \\ B(k) &= \frac{1}{2}\left(1-\frac{i\alpha k^2}{2\omega}\right)\frac{F(k)}{\sqrt{2\pi}} \\ \omega = \omega(k) &= \sqrt{c^2k^2 - \frac{\alpha^2k^4}{4}} \\ F(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx. \end{aligned}$$