Exercise 22

Solve the *initial-value problem* (Debnath 1994, p. 115) for the two-dimensional surface waves at the free surface of a running stream of velocity U. The problem satisfies the following equation, boundary, and initial conditions:

$$\phi_{xx} + \phi_{zz} = 0, \quad -\infty < x < \infty,\ -h \leq z \leq 0,\ t > 0,$$

$$\left.\begin{aligned} &\phi_x + U\phi_x + g\eta = -\frac{P}{\rho}\delta(x)\exp(i\omega t), \\ &\eta_t + U\eta_x - \phi_z = 0 \end{aligned}\right\} \quad \text{on } z = 0,\ t > 0,$$

$$\phi(x, z, 0) = \eta(x, 0) = 0, \quad \text{for all } x \text{ and } z.$$

[TYPO: This should be ϕ_t!]

Solution

In order for the first boundary condition to be dimensionally consistent, the first term must be ϕ_t, similar to the equation below it for η. Also, since $-h \leq z \leq 0$, we require a boundary condition at $z = -h$.

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=-h} = 0$$

Physically this condition implies that the velocity has no normal component at the bottom of the stream. The PDEs for ϕ and η are defined for $-\infty < x < \infty$, so we can apply the Fourier transform to solve them. We define the Fourier transform with respect to x here as

$$\mathcal{F}_x\{\phi(x, z, t)\} = \Phi(k, z, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \phi(x, z, t)\, dx,$$

which means the partial derivatives of ϕ with respect to x, z, and t transform as follows.

$$\mathcal{F}_x\left\{\frac{\partial^n \phi}{\partial x^n}\right\} = (ik)^n \Phi(k, z, t)$$

$$\mathcal{F}_x\left\{\frac{\partial^n \phi}{\partial z^n}\right\} = \frac{d^n \Phi}{dz^n}$$

$$\mathcal{F}_x\left\{\frac{\partial^n \phi}{\partial t^n}\right\} = \frac{d^n \Phi}{dt^n}$$

Take the Fourier transform of both sides of the first PDE.

$$\mathcal{F}_x\{\phi_{xx} + \phi_{zz}\} = \mathcal{F}\{0\}$$

The Fourier transform is a linear operator.

$$\mathcal{F}_x\{\phi_{xx}\} + \mathcal{F}_x\{\phi_{zz}\} = 0$$

Transform the derivatives with the relations above.

$$(ik)^2 \Phi + \frac{d^2 \Phi}{dz^2} = 0$$

Expand the coefficient of Φ.

$$-k^2\Phi + \frac{d^2\Phi}{dz^2} = 0$$

Bring the term with Φ to the right side.

$$\frac{d^2\Phi}{dz^2} = k^2\Phi$$

We can write the solution to this ODE in terms of exponentials.

$$\Phi(k, z, t) = A(k, t)e^{|k|z} + B(k, t)e^{-|k|z}$$

We can use the boundary condition at $z = -h$ here to figure out one of the constants. Taking the Fourier transform with respect to x of both sides of it gives us

$$\mathcal{F}_x\left\{ \frac{\partial \phi}{\partial z}\bigg|_{z=-h} \right\} = \mathcal{F}_x\{0\}.$$

Transform the partial derivative.

$$\frac{d\Phi}{dz}\bigg|_{z=-h} = 0$$

Differentiating Φ with respect to z, we obtain

$$\frac{d\Phi}{dz}(k, z, t) = A(k, t)|k|e^{|k|z} - B(k, t)|k|e^{-|k|z}.$$

Using the boundary condition, we have

$$\frac{d\Phi}{dz}\bigg|_{z=-h} = A(k, t)|k|e^{-|k|h} - B(k, t)|k|e^{|k|h} = 0 \quad \rightarrow \quad A(k, t) = B(k, t)e^{2h|k|},$$

so

$$\Phi(k, z, t) = B(k, t)[e^{-|k|z} + e^{(2h+z)|k|}]. \tag{1}$$

Take the Fourier transform with respect to x of the boundary conditions at $z = 0$ now.

$$\mathcal{F}_x\{\phi_t + U\phi_x + g\eta\} = \mathcal{F}_x\left\{ -\frac{P}{\rho}\delta(x)e^{i\omega t} \right\}$$
$$\mathcal{F}_x\{\eta_t + U\eta_x - \phi_z\} = \mathcal{F}_x\{0\}$$

Use the linearity property.

$$\mathcal{F}_x\{\phi_t\} + U\mathcal{F}_x\{\phi_x\} + g\mathcal{F}_x\{\eta\} = -\frac{P}{\rho}e^{i\omega t}\mathcal{F}_x\{\delta(x)\}$$
$$\mathcal{F}_x\{\eta_t\} + U\mathcal{F}_x\{\eta_x\} - \mathcal{F}_x\{\phi_z\} = 0$$

Transform the partial derivatives.

$$\frac{d\Phi}{dt} + U(ik)\Phi + gH = -\frac{P}{\rho\sqrt{2\pi}}e^{i\omega t} \tag{2}$$
$$\frac{dH}{dt} + U(ik)H - \frac{d\Phi}{dz} = 0 \tag{3}$$

Solve equation (2) for H.

$$H(k,t) = -\frac{1}{g}\left(\frac{P}{\rho\sqrt{2\pi}}e^{i\omega t} + Uik\Phi + \frac{d\Phi}{dt}\right)$$

Take a derivative of this with respect to t.

$$\frac{dH}{dt} = -\frac{1}{g}\left(\frac{Pi\omega}{\rho\sqrt{2\pi}}e^{i\omega t} + Uik\frac{d\Phi}{dt} + \frac{d^2\Phi}{dt^2}\right)$$

Use equation (1) to write expressions for $d\Phi/dt$ and $d^2\Phi/dt^2$.

$$\frac{d\Phi}{dt} = \frac{dB}{dt}[e^{-|k|z} + e^{(2h+z)|k|}] \quad \rightarrow \quad \left.\frac{d\Phi}{dt}\right|_{z=0} = \frac{dB}{dt}(1 + e^{2h|k|})$$

$$\frac{d^2\Phi}{dt^2} = \frac{d^2B}{dt^2}[e^{-|k|z} + e^{(2h+z)|k|}] \quad \rightarrow \quad \left.\frac{d^2\Phi}{dt^2}\right|_{z=0} = \frac{d^2B}{dt^2}(1 + e^{2h|k|})$$

The equations for H and dH/dt become (noting that $\Phi(k,0,t) = B(k,t)(1 + e^{2h|k|})$)

$$H(k,t) = -\frac{1}{g}\left[\frac{P}{\rho\sqrt{2\pi}}e^{i\omega t} + UikB(1 + e^{2h|k|}) + \frac{dB}{dt}(1 + e^{2h|k|})\right]$$

$$\frac{dH}{dt} = -\frac{1}{g}\left[\frac{Pi\omega}{\rho\sqrt{2\pi}}e^{i\omega t} + Uik\frac{dB}{dt}(1 + e^{2h|k|}) + \frac{d^2B}{dt^2}(1 + e^{2h|k|})\right]$$

Plug these two equations into equation (3) to get an ODE for $B(k,t)$. ϕ_z is obtained by differentiating equation (1) with respect to z and then setting z equal to zero.

$$-\frac{1}{g}\left[\frac{P\varepsilon}{\rho\sqrt{2\pi}}e^{i\omega t} + Uik\frac{dB}{dt}(1 + e^{2h|k|}) + \frac{d^2B}{dt^2}(1 + e^{2h|k|})\right]$$
$$- \frac{Uik}{g}\left[\frac{P}{\rho\sqrt{2\pi}}e^{i\omega t} + UikB(1 + e^{2h|k|}) + \frac{dB}{dt}(1 + e^{2h|k|})\right] - |k|(e^{2h|k|} - 1)B = 0$$

Simplifying this equation gives

$$\frac{d^2B}{dt^2} + 2Uik\frac{dB}{dt} + (g|k|\tanh h|k| - k^2U^2)B = -\frac{iP(kU + \omega)}{\rho\sqrt{2\pi}(1 + e^{2h|k|})}e^{i\omega t},$$

where the identity,

$$\tanh h|k| = \frac{e^{2h|k|} - 1}{e^{2h|k|} + 1},$$

was used. The solution to this second-order inhomogeneous ODE is

$$B(k,t) = C_1(k)e^{-it(Uk + \sqrt{g|k|\tanh h|k|})} + C_2(k)e^{-it(Uk - \sqrt{g|k|\tanh h|k|})}$$
$$+ \frac{iP(kU + \omega)e^{i\omega t - h|k|}}{2\sqrt{2\pi}\rho[(kU + \omega)^2\cosh h|k| - g|k|\sinh h|k|]}.$$

Make use of the initial conditions in order to determine $C_1(k)$ and $C_2(k)$. First take the Fourier transform of both sides of them.

$$\phi(x,z,0) = 0 \quad \rightarrow \quad \mathcal{F}_x\{\phi(x,z,0)\} = \mathcal{F}_x\{0\}$$
$$\Phi(k,z,0) = 0 \tag{4}$$
$$\eta(x,0) = 0 \quad \rightarrow \quad \mathcal{F}_x\{\eta(x,0)\} = \mathcal{F}_x\{0\}$$
$$H(k,0) = 0 \tag{5}$$

Applying equation (4) yields

$$\Phi(k,z,0) = B(k,0)[e^{-|k|z} + e^{(2h+z)|k|}] = 0 \quad \to \quad B(k,0) = 0,$$

which means

$$B(k,0) = C_1(k) + C_2(k) + \frac{iP(kU+\omega)e^{-h|k|}}{2\sqrt{2\pi}\rho[(kU+\omega)^2\cosh h|k| - g|k|\sinh h|k|]} = 0.$$

Solve this for $C_1(k)$.

$$C_1(k) = -C_2(k) - \frac{iP(kU+\omega)e^{-h|k|}}{2\sqrt{2\pi}\rho[(kU+\omega)^2\cosh h|k| - g|k|\sinh h|k|]}$$

Earlier we solved equation (2) for $H(k,t)$. This will be the equation we use to determine $C_2(k)$.

$$H(k,0) = -\frac{1}{g}\left[\frac{P}{\rho\sqrt{2\pi}} + Uik\Phi(k,0,0) + \frac{d\Phi}{dt}(k,0,0)\right] = 0$$

The left side yields a very ugly expression involving $C_2(k)$ that can nevertheless be solved. Now that $C_1(k)$ and $C_2(k)$ are solved for, $\Phi(k,z,t)$ is known and, consequently, $H(k,t)$ is as well. The final expressions are as follows.

$$\begin{aligned}\Phi(k,z,t) = {} & \frac{Pe^{-iUkt-|k|(2h+z)}(1+e^{2(h+z)|k|})}{4\sqrt{2\pi}\rho[(Uk+\omega)^2\cosh h|k| - g|k|\sinh h|k|]}\times \\ & \left\{2ie^{h|k|}(Uk+\omega)\left[e^{it(Uk+\omega)} - \cos(t\sqrt{g|k|\tanh h|k|})\right]\right. \\ & \left. - i\sqrt{1-e^{4h|k|}}\sqrt{g|k|}\operatorname{sech} h|k|\sin(t\sqrt{g|k|\tanh h|k|})\right\} \\ N(k,t) = {} & \frac{P\sqrt{|k|}e^{-iUkt}\sqrt{\tanh h|k|}}{i\rho\sqrt{g}\sqrt{2\pi}[-(1+e^{2h|k|})(Uk+\omega)^2 + (-1+e^{2h|k|})g|k|]}\times \\ & \left\{\sqrt{1-e^{4h|k|}}\sqrt{g|k|}\left[-e^{it(Uk+\omega)} + \cos(t\sqrt{g|k|\tanh h|k|})\right]\right. \\ & \left. - (1+e^{2h|k|})(Uk+\omega)\sin(t\sqrt{g|k|\tanh h|k|})\right\}\end{aligned}$$

All that's left is to take the inverse Fourier transform of Φ and H to get ϕ and η.

$$\phi(x,z,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(k,z,t)e^{ikx}\,dk \quad \text{and} \quad \eta(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}H(k,t)e^{ikx}\,dk$$